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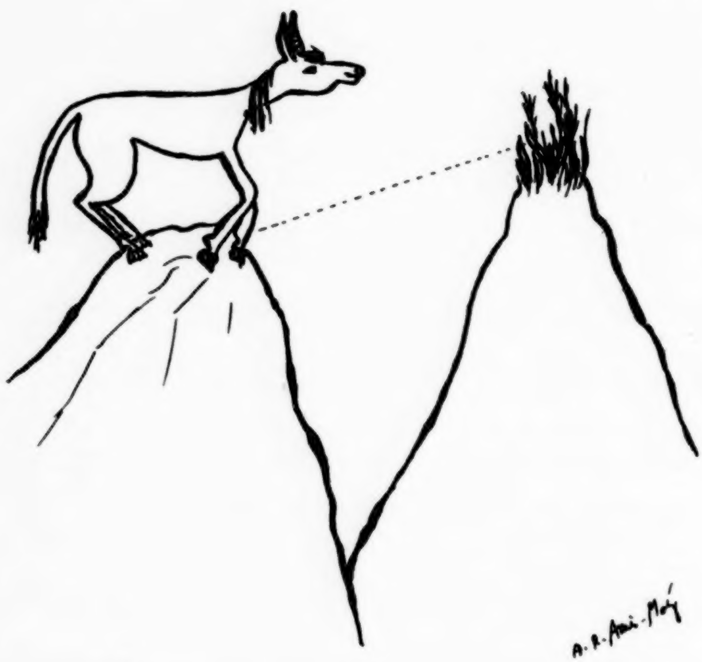
Relative & Absolute



Straight line is the shortest, but not necessarily the qickest way to get there!

SOME COMMENTS ON THE ROLE OF THE AXIOM OF CHOICE IN THE DEVELOPMENT OF ABSTRACT SET THEORY*

William Leonard Zlot

The axiom of choice asserts that if M is a set of non-null, disjoint sets m_i, then there exists a set S, called a set of choice, which contains one and only one element from each of the m_i. The existence of such a collection as S seems assured on the basis of intuitive reasoning. However, it must be recalled that the word *set* is undefined, and only attains meaning through its occurrences in the axioms of set theory.

An equivalent assertion is that of the well-ordering theorem which states that every class can be so ordered that each of its non-empty subclasses has a first element. This article gives a brief description of the significance of the axiom of choice and the well-ordering theorem in the development of abstract set theory.

The birth of Cantor's theory of sets. The use of the procedure of cuts by Dedekind [11] to define irrational numbers made it clear that the real numbers could only be attained (conceptually) by conceiving of an actual infinity of numbers previously introduced (rationals). Thus, it became necessary to admit reasoning which dared to use an actual infinity of propositions as premises, each corresponding in a one-to-one fashion to the rationals [22].

Many of the predecessors of Cantor had tried to reason on the infinite, but so many contradictions were met that the actual infinite was relegated to a place outside of mathematics. Cantor taught mathematicians to consider certain distinctions such as that between cardinal and ordinal, which only the grammarians had previously observed. Cantor also pointed out that, although there is no real distinction in the case of finite collections, it becomes necessary to distinguish between these conceptions in the case of infinite aggregates [4]. It can be shown that it is by means of the well-ordering theorem (or, equivalently, the axiom of choice) that the concepts of cardinal and ordinal are united for the infinite case too.

Many difficulties arose when analysis was made of objects whose

*This paper is an excerpt from the writer's Doctoral Thesis written at Columbia University in 1957. The complete title of the thesis is *The Role of the Axiom of Choice in the Development of the Abstract Theory of Sets*, Library of Congress number Mic 57-2164.

nature was relatively obscure. For example, consider the well-ordered sequence $1, 2, \cdots, n, \cdots, \omega, \cdots$. If one wants to demonstrate a proposition for all the elements of the collection, the ordinary reasoning by recurrence (mathematical induction) does not suffice, for it does not permit the attainment of ω, since it has no immediate predecessor. Cantor appropriately extended the method of reasoning employed in mathematical induction so that one would be allowed to assert the truth of a proposition if it was proved for all the ordinal numbers which precede any given ordinal. This process is known as transfinite induction, and was first used by Cantor in theorems relating to exponentiation by infinite ordinals [5]. When this procedure is permitted, the theory of sets can be developed with the same simplicity and beauty as that of the ordinary integers. However, it can be shown that one can permit the extension of this type of reasoning to the theory of sets in general when and only when the well-ordering theorem (or, equivalently, the axiom of choice) is assumed to be valid. Specifically, it can be proved that in order for the principle of transfinite induction to be applicable to an ordered set, it is necessary and sufficient that the set be well-ordered. Incidentally, it may be pointed out that Gentzen proved the consistency of ordinary arithmetic by using transfinite induction [17].

There also has been much research into the extension of the various rules of calculation of ordinary arithmetic to cover the case where any or all of the factors may be infinite cardinal or ordinal numbers. Here, again, one very often sees the seeming indispensability of the axiom of choice in permitting the extension [24]. It is interesting to observe, on the other hand, that many theorems of ordinary arithmetic hold for transfinite numbers even if the axiom of choice is not assumed [1].

The axiomatization of set theory. One of the most striking features of the principle of choice is that the necessity for its assumption is a priori completely unexpected. It had been taken for granted by mathematicians, including Cantor. It was Zermelo who first explicitly mentioned the necessity for its use [30].

One may draw an analogy between the theory of sets and geometry in that both are based upon observations from the physical world. The first application of the axiomatic method may be found in the field of geometry [12] and was motivated by the failure of man's intuition to answer all of the geometrical questions that occurred to him, i.e., problems relating to the parallel-postulate. Mathematicians such as Lobachewsky, Bolyai, Hilbert, etc., found it advantageous to apply the axiomatic method to this discipline which had been born from the concrete world. This is similar to the situation in the theory of sets. The antinomies and the problem of the axiom of choice were primary motivations for the axiomatizations of Zermelo and his successors. Thus, a branch of mathematics which was

based upon the most fundamental and elementary concepts i.e., objects, collections, counting, etc., came to be viewed in a different light.

The axiom of choice, so unanticipated, and yet so powerful, was one of the primary motivations for the axiomatizations. For, consider the following idea. The axiom of choice is a statement which captures in a finite number of words much of the spirit of Cantor's genetic extension of number and of the concept of an infinity of operations. The natural reaction of researchers, such as Zermelo, was to try to find as small a number of statements as possible that would describe all of set theory. Now, since the very discovery of the need for the explicit postulation of the principle of choice came as a shock and surprise to mathematicians, the curiosity of the researchers was quite properly aroused concerning the possibility that other unexpected rules might have to be set down in order to yield Cantor's "naïve" set theory.

Certainly, the fact that the axiom of choice is independent of the other axioms is interesting in itself. In fact, Church denied the axiom of choice, and obtained results which seem to be non-contradictory, including the theorem that the continuum cannot be well-ordered (within the framework of his assumptions) [7]. The apparent need to employ intuitively unreasonable objects such as *non-sets* [16] and *irregular sets* [26] to prove independence hints at the peculiar nature of the axiom of choice.

Of course, Gödel's consistency proof, relative to that of the remaining system of axioms, is very satisfying to those who advocate liberal use of the axiom of choice because one can enjoy the myriad of advantages which this proposition provides without fearing the addition of new danger-points. Many mathematicians would take this result as reason enough to introduce the axiom and reap advantages of its free use.

The nature of the axiom of choice. The very nature of the axiom of choice was a prime motivation for the development of the group known as the Intuitionists. Fraenkel states that the axioms of choice and infinity both raise certain difficulties since it is not clear to what extent they belong to "true facts" of evidence, or tautologies [14]. Ramsey stated that :

> The axiom of choice is a tautology and must, therefore, be provable, for it does not seem the least unlikely that there should be a tautology which could be stated in finite terms, whose proof was nevertheless infinitely complicated, and therefore, impossible for us [27].

Now, for the Intuitionists the word *existence* is equivalent to constructibility, that is, mathematics is visualized as "a world of constructions to be performed in time [14]." On the other hand, the view may be held that the word *existence* need not be defined; to *exist* means to *exist* [28].

The two opposing views reflect two different feelings towards

mathematics. The Intuitionists, empiricists, realists, or whatever one may call them, think that the solidity of mathematics is a more important consideration than that of expansion. The view of the so-called idealists may be described as one which advocates the extension of the limits of mathematics – so long as no inconsistencies are introduced.

Fraenkel says that the problem does not involve the question of the construction of a set of choice since the axiom does not determine the set in question in a unique manner [13]. Fraenkel further states that the problem is not to be enunciated, "Is it possible to give a set of choice?" but rather: "Should one deny the non-existence of sets of choice?" Cavaillès said that the main attacks on the axiom involve two basic arguments. (1) The axiom does not give the means of passing from the set M of sets m_i to the choice set, and (2) the axiom does not characterize the choice-set in a unique fashion. Cavaillès stated that the axioms of the power-set and of the union, both of which are acceptable to most mathematicians, possess the first characteristic. Therefore, only the second argument remains against the axiom [6].

Therefore, the well-ordering theorem, the theorem of general comparability, and other related assertions, all have the same purely existential character as the principle of choice. In other words, the principle may be asserted at the beginning of a discussion, logical conclusions may be drawn, and no thought need be given to questions of attainability.

Thus, all of axiomatic set theory, if the principle of choice or its logical equivalent is asserted, must be visualized as a purely existential structure. Therefore, one of the main consequences of the acceptance of the axiom of choice is the fact that it has helped to mold man's view of the abstract theory of sets.

At this point it may be noted that several paradoxes have been derived by use of the axiom of choice [3]. Lévy states, however, that the rejection of the axiom of choice leads to even more paradoxical consequences – consequences which can be admitted only by one who doubts human reason [23].

Transfinite number theory. In conclusion, it may be pointed out that the assumption of the axiom of choice allows man to consider the abstract theory of sets as a modified and extended form of number theory. In order to comprehend this role of the axiom of choice the following facts should be noted.

Hankel stated that any extension of the natural number domain to greater and greater complexity must be dominated by a guiding and unifying idea which he called *the principle of permanence of formal operations* (Prinzip der Permanenz formaler Gesetze). Hankel expressed his principle as follows:

> Equal expressions represented in the general symbols of universal arithmetic are to remain equal if the symbols cease to denote simple quantities (Grössen), and, therefore, also if the interpretation of the operations is altered [18].

An example of the principle is the fact that $ab = ba$ is to remain valid when a, b are complex. Dantzig interprets the principle of permanence as follows [10]: A collection of an infinity of symbols is called a *number field*, and each element in it is called a *number* if the following three conditions, collectively entitled the principle of permanence, are applicable to this collection:

(1) The sequence of natural numbers can be identified among the elements of the collection.

(2) Criteria of rank can be established which will permit one to determine for any two elements whether they are equal, or if not equal, which is greater, these criteria reducing to the natural criteria when the two elements are natural numbers.

(3) A scheme of addition and multiplication can be defined for any two elements of the collection which will have the commutative, associative, and distributive properties of the natural operations bearing these names.

Certainly, condition (1) is satisfied in the abstract theory of sets because the class of natural numbers is simply the first number-class. Condition (2) is valid in the abstract theory of sets if and only if the axiom of choice is assumed. It is true that condition (3) is not completely satisfied because, for example, addition and multiplication of ordinal numbers are not commutative, and only left-hand distributivity, that is, $a(b+c) = ab+ac$, is provable, however, a great number of the familiar propositions of the theory of numbers are preserved if and only if the axiom of choice is assumed. It is interesting to note that Hankel, himself, stated that the principle of permanence can be extended and changed somewhat in order to facilitate the extension of the number system [18]. Hankel cited the denial of the commutative law of multiplication as an example of such a modification.

In recent years a considerable number of investigations have been undertaken with the intention of discovering propositions which are equivalent to the axiom of choice [2]. The desire to find theorems which are equivalent to the axiom of choice stems not only from the great natural interest that mathematicians have in the discovery of relationships between propositions but also from the attitude which some mathematicians have towards the axiom of choice. For example, those mathematicians who suspect the validity of the axiom of choice prefer to use an equivalent proposition if they feel it to be more intuitively plausible than the axiom of choice. Obviously, such mathematicians do not fully understand the

nature of a logical equivalence, for the use of a proposition which is equivalent to the axiom of choice implicitly sanctions the admissibility of an infinity of choices.

Contemporary research in abstract set theory has also consisted of investigations which are similar to those of Fraenkel [15] in that they involve the imposition of restrictions on either the cardinality of M (the set of mutually disjoint non-empty sets m_i) or on each of the m_i. In particular, writers such as Szmielew [29] investigated certain necessary and sufficient conditions for the case where each of the m_i is finite.

In recent years the most prevalent type of approach to problems related to well-ordering and the axiom of choice is through the methods of symbolic logic – specifically, the work of Church [8, 9] and Kleene [19, 20, 21] on recursive functions. Markwald states that the motivation for research on the theory of recursive functions and of *constructive* building is that the classical theory of ordinals contains many non-constructive methods, which, in the extreme, have led to paradoxes [25]. A *constructive* or *effective* operation on a denumerable set is one for which a fixed set of instructions can be chosen so that, for each of the infinitely many objects of the set, the operation can be completed by a finite process in accordance with the instructions.

As an illustration of a typical problem that can be raised in connection with the notion of constructive or effective operations consider a question that Kleene raised: How far can a system of formal notations for the ordinal numbers be extended into the second-class? In other words, what is the limit of man's ability to tag or designate the numbers of the second-class? Kleene proved that there is an ordinal number ω_1 of the second-class such that there are systems of notation which extend to all ordinals less than ω_1 but none in which ω_1 itself is assigned a notation.

Future research in set theory will probably follow two courses. The first involves investigations of the foundations of set theory – an approach which is guided by the results of mathematical logic and metamathematics. This approach entails the analysis of the methods and limits of mathematics. The second course involves the continued and further application of set theory to the other branches of mathematics and indeed to science, in general. These courses reflect two different types of scientific minds. One is ever probing into the depths and sources of a discipline; and the other eagerly looks towards the extension and generalization of previous results.

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THE PROBABILITY THAT THE ROOTS OF A REAL QUADRATIC EQUATION LIE INSIDE OR ON THE CIRCUMFERENCE OF THE UNIT CIRCLE IN THE COMPLEX PLANE

Joseph W. Andrushkiw

Introduction. In connection with problems of this kind usually the equation $x^2+px+q=0$ is considered. The results obtained are paradoxical (American Math. Monthly, vol. 60, 1953, p. 553 E 1054; J. L. Coolidge, An Introduction to Mathematical Probability, p. 79; H. Levy & L. Roth, Elements of Probability, p. 84-85) or the intervals of variation of coefficients p, q are artificial (The Mathematical Gazette, vol. xxxix, 1955, Math. Note No. 2485). In this note we consider the equation $xz^2+wz+y=0$. The results obtained give the probability of real and imaginary roots. By the help of the same method the following problems can also be solved: 1. Find the probability that the roots of a real quadratic equation are real and in the interval (a, b), 2. Find the probability that the roots are imaginary and their real parts are in the interval (a, b).

I.

Consider the coefficients x, y, w of

(1) $$xz^2+wz+y=0$$

as the rectangular cartesian coordinates of a point P. If P lies outside or on the surface of the cone

(2) $$F(x, y, w)=w^2-4xy=0,$$

equation (1) has real roots. Further, it is easy to see that the cone (2) is the envelope of the family of planes (c as parameter)

(3) $$f(x, y, w, c)=c^2x+cw+y=0,$$

and

(4) $$x=\frac{y}{c^2}=-\frac{w}{2c}$$

is equation of the line of tangency of the plane (3) and the cone (2) (Fig. 1.: the cone $OND_2N_1M_1DMN$; tangent planes OA_1T_1, OB_1T_1, $OA_2P_2T_2$, $OB_2R_2T_2$, $OA_3S_3P_3K_3T_3$, $OB_3Z_3R_3L_3T_3$ etc; the corresponding lines of tangency are: OA_1, OB_1, OA_2, OB_2, OA_3, OB_3, etc).

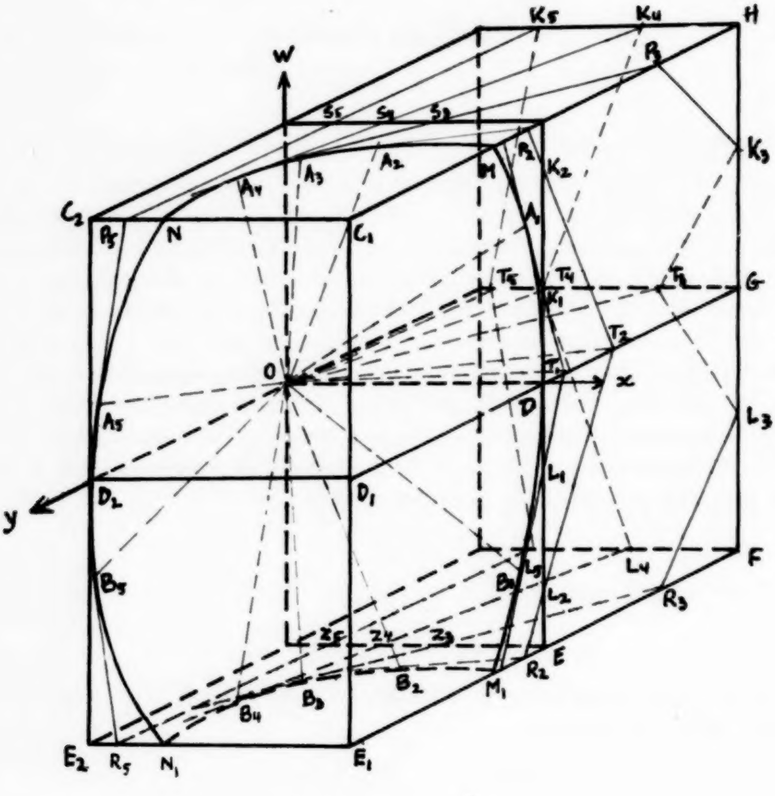

Fig.1.

If x, y, w are the coordinates of a point that lies in the plane (3) equation (1) has c for its real root. It follows that equation (1) has for its (real) roots c and $-c$ if and only if x, y, w are the coordinates of a point which lies on the line of intersection of the planes

(5) $$c^2x + cw + y = 0, \qquad c^2x - cw + y = 0.$$

Consequently, equation (1) with restricted coefficients

$$-k \leqq x \leqq k, \quad -k \leqq y \leqq k, \quad -k \leqq w \leqq k \tag{6}$$

has both its roots real and in the interval $(-c, c)$ if and only if x, y, w are the coordinates of a point which lies inside the cube (6), outside the cone (2) and between the tangent planes (5). E.g. in Fig. 1 OA_1T_1 and OB_1T_1 represent two tangent planes for $|c|<\frac{1}{2}$ (in the right half-space), and OA_1DB_1 represents the portion of the conical surface. For all values x, y, w which are the coordinates of the points in the portion of the cube $OA_1T_1B_1DA_1$ equation (1) has its roots real and in the interval $(c, -c)$.In the left half-space there is another symmetrical portion of the cube such that for the coordinates of its points equation (1) has real roots contained in $(c, -c)$.

Denoting by $2v$ the volume of the above portions of the cube, and by $P_r(c)$ the probability that the equation (1) with restricted coefficients (6) has its roots real and in the interval $(c, -c)$, we define

$$P_r(c) = \frac{2v}{8k^3} = \frac{v}{4k^3}. \tag{7}$$

If c varies from 0 to ∞, the tangent plane (3) revolves in a clockwise direction around the cone (2). For $c = \infty$ it coincides with the yw coordinate plane and for $c = 0$ with the xw coordinate plane.

The cube (6) cuts from the cone (2) in the right half-space the solid bounded by the plane sides $NC_1D_1E_1N_1D_2N$, $C_1D_1E_1M_1DMC_1$, NMC_1, $N_1M_1E_1$ and the conical surface $ONMM_1N_1$ (Fig. 1). A similar solid is cut by the cube in the left half-space (in the 3rd and 7th octants).

The curves ND_2N_1 and MDM_1 are parabolas whose equations are

$$w^2 = 4kx, \quad w^2 = 4ky, \tag{8}$$

respectively. The curves NM and N_1M_1 are hyperbolas with equations

$$4xy = k^2. \tag{9}$$

The solid formed by the tangent planes, conical surface and the sides of the cube has different shape depending upon the value of c. In Fig. 1 are shown all 5 fundamental positions:

$$\begin{array}{lll} & 1.\ 0 < c \leqq 1/2, & \text{the solid } OA_1K_1T_1L_1B_1DA_1 \\ & 2.\ 1/2 \leqq c \leqq 1, & \text{the solid } OA_2P_2K_2T_2L_2R_2B_2M_1DMA_2 \\ (10) & 3.\ 1 \leqq c \leqq (\sqrt{5}+1)/2, & \text{the solid } OA_3S_3P_3K_3T_3L_3R_3Z_3B_3M_1DMA_3 \\ & 4.\ (\sqrt{5}+1)/2 \leqq c \leqq 2, & \text{the solid } OA_4S_4K_4T_4L_4Z_4B_4M_1DMA_4 \end{array}$$

5. $2 \leqq c < \infty$, the solid $OA_5P_5S_5K_5T_5L_5Z_5R_5B_5N_1M_1DMNA_5$.

On computing the volume v of the solids (10) one obtains:

1. $v = (2ck)^3/18$, $P_r(c) = 8c^3k^3/72k^3 = c^3/9$.

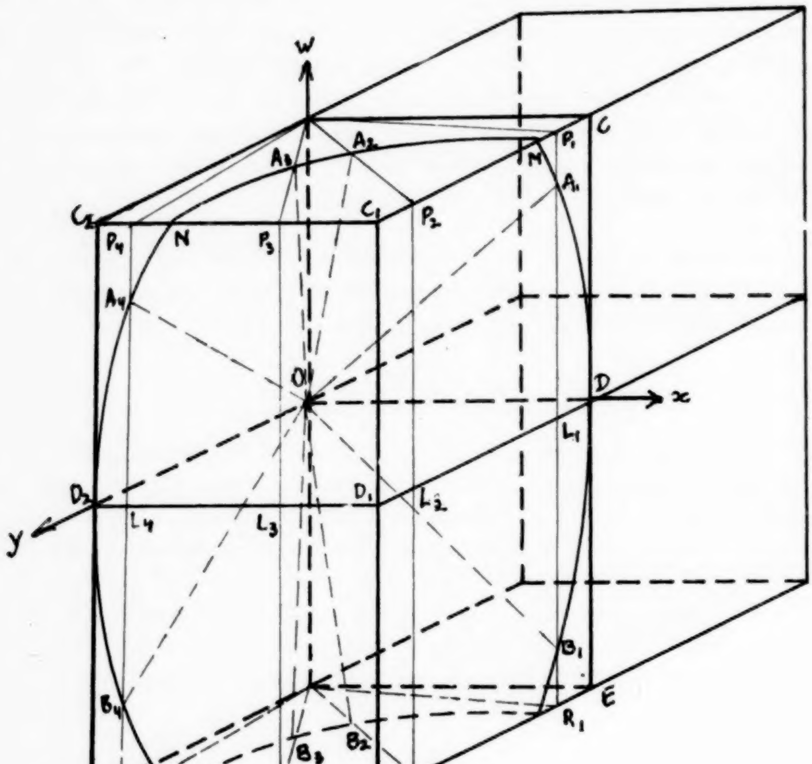

Fig.2.

Since $P_r(c)$ is expressed in a form independent of k, it may be assumed for the probability of real roots in the interval $(-c, c)$ of equation (1) with unrestricted coefficients. Similar computations yield

2. $P_r(c) = 11/144 + (\log 2c)/24 + (c^2 - c)/4$

3. $P_r(c) = 11/144 + (\log 2c)/24 - (1/12c^3)(c^6 - 3c^5 + 3c^2 - 1)$

$$4.\ P_r(c) = 71/144 + (\log 2c)/24 - (c+1)/4c^2$$

$$5.\ P_r(c) = 41/72 + (\log 2)/12 - (9c^2+2)/18c^3.$$

As a particular result one finds $P_r(\infty) = 41/72 + (\log 2)/12 = 62.72\cdots\%$, i.e. among 10000 random real quadratic equations one may expect 6272 equations with real roots.

II.

The imaginary roots of equation (1) have their absolute value not greater than $c > 0$, if and only if x, y, w are the coordinates of a point which lies inside the cone (2) and to the right (in the right half-space) of the plane

$$y = c^2 x \tag{11}$$

(e.g. in Fig. 2 the plane $OA_1L_1B_1$ for $c < 1/2$). Namely, in case of imaginary roots, the absolute value of the roots is

$$|(-w + i\sqrt{4xy - w^2})/2x| = \sqrt{x/y}\,; \tag{12}$$

hence, from $\sqrt{y/x} \leqq c$ follows $y \leqq c^2 x$. In the third and seventh octants there is another symmetrical portion of the cube such that for the coordinates x, y, w of its points equation (1) has imaginary roots with absolute value not greater than c.

The probability $P_i(c)$ that equation (1) has imaginary roots absolutely not greater than c is defined as the ratio

$$P_i(c) = 2v/8k^3 = v/4k^3 \tag{13}$$

where v is the volume of that portion of the cube which lies between the plane (11) and the conical surface to the right of the plane. According to the values of c four fundamental positions have to be distinguished. They are indicated in Fig. 2 by the solids:

$$\begin{aligned}
&1.\ OA_1DB_1L_1A_1,\ 0 < c \leqq 1/2,\\
&2.\ OA_2MDM_1R_2B_2O,\ 1/2 \leqq c \leqq 1,\\
&3.\ OA_3MDM_1E_1R_3B_3O,\ 1 \leqq c \leqq 2,\\
&4.\ OA_4NMDM_1E_1N_1B_4A_4,\ 2 \leqq c < \infty.
\end{aligned} \tag{14}$$

Computing the volumes of the above solids (14) the following results are obtained:

$$(15)\qquad \begin{aligned} &1.\ P_i(c) = 2c^3/9,\\ &2.\ P_i(c) = (36c^2-5-6\log 2c)/144,\\ &3.\ P_i(c) = (67-6\log 2c)/144 - 1/4c^2,\\ &4.\ P_i(c) = (31-6\log 2)/72 - 2/9c^2. \end{aligned}$$

III.

The solids whose volumes are computed in the case of real or imaginary roots have only their boundaries in common. Therefore, the probability $P(c)$ that a quadratic equation has roots (real or imaginary) with absolute values not greater than c may be defined by

$$(16)\qquad P(c) = P_r(c) + P_i(c).$$

Hence the answer to our problem is $P(1) = 7/24 = 29.17\cdots\%$.

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ON ISOSCELES ORTHOGONALITY

Harry F. Davis

Any student of vector algebra can easily prove that two vectors v and w are orthogonal (perpendicular) if and only if the length of $v+w$ equals the length of $v-w$. This is the point of digression of the present note.

In advanced courses in differential equations, students become acquainted with the idea of a *norm*, which is a generalization of length. The ordinary length $\sqrt{x^2+y^2}$ of a vector in the plane may be called its *Euclidean norm* to distinguish it from other norms, such as $|x|+|y|$. Relative to other norms, there is, in general, no notion comparable to orthogonality that is nearly so useful. The following observations are related to this fact; we have found them to be interesting to students of modern algebra, and not well known to colleagues.

Let V denote a vector space with real scalars, equipped with a norm $||v||$. If there exists an inner product (v,w) in V such that $\sqrt{(v,v)}$ equals the norm $||v||$ for all v in V, then V is said to be Euclidean. In this case $(v,w)=0$ if and only if $||v+w|| = ||v-w||$. Thus it is possible to define orthogonality entirely in terms of the norm, and this suggests the following definition, which is due to R. C. James [3]. Vectors v and w are said to be *isosceles orthogonal* if $||v+w|| = ||v-w||$.

Call the collection $C(v)$ of all vectors isosceles orthogonal to a vector v the *orthogonal complement* of v. If V is Euclidean, isosceles orthogonality is entirely equivalent to ordinary orthogonality, and in this case $C(v)$ is a subspace for every v. This simple observation is very useful for showing that there can exist no inner product giving rise to the norm $|x|+|y|$ in the plane. For let $u=(2,1)$, $v=(-1,1)$, and $w=(1,-2)$; then v and w are isosceles orthogonal to u (in this norm), but $v+w$ is not. Similar examples are easily constructed for other norms, such as $\max(|x|,|y|)$, and this observation also provides an easy way to demonstrate that certain normed function spaces are not Hilbert spaces.

THEOREM 1. *A real normed vector space V is Euclidean if and only if $C(v)$ is a subspace, for every element v in V.*

THEOREM 2. *A two dimensional real normed vector space W is isometric with the Euclidean plane if and only if the orthogonal complement of every vector is a subspace.*

Theorem 1 may be proved using a stronger theorem due to R. C. James [3]. His theorem depends on a theorem due to Ficken [2] the proof of which

makes use of a theorem of Jordan and von Neumann [4]. We give a simple direct proof of the theorem, using a device from [4] to show that, if Theorem 2 is valid, then Theorem 1 must be also.

As remarked above, the condition given in Theorem 1 is clearly *necessary*. To prove the condition is *sufficient*, assume that $C(v)$ is a subspace for every v in V. If the dimension of V is 0 or 1, the theorem is trivial; otherwise define (v,w) to be $(||v+w||^2 - ||v-w||^2)/4$. Given any vectors v and w, there exists a two dimensional subspace W containing v and w. Assuming that Theorem 2 is valid, W must be isometric with the Euclidean plane, and this isometry implies that $(cv,w) = c(v,w)$ for all real numbers c. The other properties of the inner product are easily verified, except for additivity, which is harder since three vectors need not be coplanar. The isometry implies $||f+g||^2 + ||f-g||^2 = 2(||f||^2 + ||g||^2)$ for any pair of vectors f and g. In this expression let $f = \frac{1}{2}(v+w) \pm u$ and $g = \frac{1}{2}(v-w)$, subtract, and use the definition of (v,w) to prove that $(v,u) + (w,u) = 2(\frac{1}{2}(v+w),u)$, which equals $(v+w,u)$ by letting $c = 2$ above.

This reduces the proof of Theorem 1 to that of Theorem 2, which is accomplished by three simple lemmas, in each of which is assumed the hypothesis that orthogonal complements in W are always subspaces. It is convenient to identify the elements of W with ordered pairs of real numbers (x,y) by using any fixed basis in W.

LEMMA 1. *If v is a nonzero vector, $C(v)$ is a one dimensional subspace of W; in particular, $C(v)$ contains at least one vector of unit norm.*

PROOF. If v is nonzero, v is obviously not an element of $C(v)$, so $C(v) \neq W$. This implies $C(v)$ has dimension less than two. We prove the existence of a unit vector isosceles orthogonal to v. An easy argument shows that the norm in W is a continuous function of x and y. If $v = (x,y)$, let $g(t) = (x\cos t - y\sin t,\ y\cos t + x\sin t)$, so that $g(0) = v$, $g(\pi) = -v$, and $g(t) \neq (0,0)$ for $0 \leq t \leq \pi$. Then $h(t) = ||v+g(t)|| - ||v-g(t)||$ is a continuous real valued function in the interval $0 \leq t \leq \pi$. Since $h(0)$ is positive and $h(\pi)$ is negative, we have $h(s) = 0$ for some s in this interval, and the desired unit vector is then $g(s)/||g(s)||$.

LEMMA 2. *There exists an inner product (u,v) in W with the property that any two vectors u and v are isosceles orthogonal if and only if $(u,v) = 0$.*

PROOF. Let f be any vector with unit norm, and g a unit vector isosceles orthogonal to f (it exists, by Lemma 1). Define the inner product (u,v) as usual, using the new coordinate system in which $f = (1,0)$ and $g = (0,1)$. The norm in W coincides with the Euclidean norm (defined through the new coordinates in the usual way) for vectors having one coordinate equal to zero. Since f and g are isosceles orthogonal, $||af+bg|| = ||af-bg||$, since f in $C(g)$, g in $C(f)$ implies af is in $C(g)$ and bg is in $C(f)$. Hence

$||(a,b)|| = ||(a,-b||$, and a similar argument (using the isosceles orthogonality of $(1,1)$ and $(1,-1)$ instead) shows that the norm of a vector of the form (a,b) equals the norm of (b,a). Hence $||(x-y,y+x)|| = ||(y+x,x-y)|| = ||(y+x,y-x)||$ implying that vectors of the form (x,y) and $(-y,x)$ are isosceles orthogonal. Hence so also are vectors of the form $r(x,y)$ and $s(-y,x)$, proving that vectors which are orthogonal relative to the new coordinate system are isosceles orthogonal.

The converse is proved as follows. If u and v are isosceles orthogonal nonzero vectors, let v' be a nonzero vector orthogonal to u. Then v' is, by the above, isosceles orthogonal to u, and is in $C(u)$ along with v, so v and v' are multiples, proving that u and v are orthogonal.

LEMMA 3. *Perpendicular reflection in any line in W passing through the origin is a norm-preserving automorphism.*

PROOF. Any vector can be written in the form $u+v$, where u is in the given line and v is orthogonal to the line. Then the image of $u+v$ under reflection in the line is $u-v$, and $||u+v|| = ||u-v||$ since orthogonal vectors are isosceles orthogonal by Lemma 2.

PROOF OF THEOREM 2. Any rotation of vectors in the plane is the composite of two reflections, each of which preserves norm by Lemma 3, so the norm of any vector on the circle $x^2+y^2=1$ equals that of $(1,0)$, i.e. unity, and hence the norm equals the Euclidean norm everywhere.

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NOTE ON THE COEFFICIENTS OF $\cosh x/\cos x$

L. Carlitz

In a recent paper in the Mathematics Magazine [1], J. M. Gandhi has defined a set of rational integers S_{2n} by means of

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!} \tag{1}$$

and conjectured that

$$S_{2n} = 2^n S'_{2n}, \tag{2}$$

where S'_{2n} is odd. In this note we show that this conjecture is correct and moreover that

$$S'_{2n} \equiv (-1)^{\frac{1}{2}n(n-1)} \pmod 4. \tag{3}$$

Differentiating (1) we get

$$\frac{\cosh x}{\cos x}(\tan x + \tanh x) = \sum_{n=1}^{\infty} S_{2n} \frac{x^{2n-1}}{(2n-1)!}. \tag{4}$$

But

$$\tan x + \tanh x = -2 \sum_{r=0}^{\infty} C_{4r+1} \frac{x^{4r+1}}{(4r+1)!}, \tag{5}$$

where [2, p. 28]

$$C_{2n-1} = \frac{2^{2n}(1-2^{2n})B_{2n}}{2n} \tag{6}$$

and B_{2n} denotes the Bernoulli number in the even suffix notation. Comparing (4) and (5) we get

$$S_{2n} = -2 \sum_{0 \leq 4r < 2n} \binom{2n-1}{4r+1} C_{4r+1} S_{2n-4r-2},$$

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ECCENTRICITY IN ELLIPSES

L. A. Kenna

A very intriguing relationship exists between the eccentricity of an ellipse and the angle of the plane forming the ellipse. A theorem will be formulated concerning the eccentricity of ellipses formed by planes cutting cylindrical surfaces. A proof will then be presented to substantiate the theorem.

The usual way of obtaining ellipses is by conic sections. A convenient way to illustrate this is by means of simple projection.

The following device is sometimes referred to as a "shadowgraph." A plane source of light represents the cylindrical surface. The cutting plane is represented by some transparent material. A circle drawn with opaque ink on the transparent material casts a shadow of light. An opaque surface is placed a convenient distance away from the source of light. Between the light source and the opaque plane the cutting plane is placed. When the opaque plane and the cutting plane are parallel the circle projects into a circle. At any other angle, θ, where

$$0^{0} \leqq \theta < 90^{\circ} \tag{1}$$

the circle projects into an ellipse.

The shadowgraph should be of interest to the less experienced mathematician as well as the experienced geometer. It is not a new device, but one which is very useful for graphic illustration of mathematical phenomenon and techniques.

The following is an informal proof that ellipses are formed by planes cutting right circular cylinders.

The equation of a right circular cylinder is;

$$X^2 + Y^2 = R^2 \tag{2}$$

Equation (2) assumes the cylinder is at the origin with the axis coincident with the Z-axis. Rather than finding equations of planes intersecting the cylinder, it was deemed more appropriate to rotate the axes about the Y-axis. This is the same as rotating the cylinder.

An ellipse is formed by rotating the axes through an angle θ. Equation (2) no longer represents the cylinder with respect to the new axes, X' and Z'. The direction cosines of the new coordinate axes have to be found. From this it follows that,

$$X = X'\text{Cos}\, a_1 + Z'\text{Cos}\, a_3 \tag{3}$$

$$Y = Y'\text{Cos}\, b_2 = Y' \tag{4}$$

The equation of the cylinder with respect to the new axes is obtained by substituting equations (3) and (4) into equation (2). The result is :

$$X'^2\,\text{Cos}^2 a_1 + 2X'Z'\text{Cos}\, a_1\,\text{Cos}\, a_3 + Z'^2\,\text{Cos}^2 a_3 + Y'^2 = R^2 \tag{5}$$

The ellipse can now be s een by allowing the $X'Y'$-plane to be the cutting plane, while the XY-plane is the normal plane. By letting $Z' = 0$ in equation (5) the desired cutting plane is obtained which results in :

$$X'^2\,\text{Cos}^2 a_1 + Y'^2\,\text{Cos}^2 b_2 = R^2 \tag{6}$$

Dividing equation (6) by R^2 reveals the ellipse :

$$\frac{X'^2\,\text{Cos}^2 a_1}{R^2} + \frac{Y'^2\,\text{Cos}^2 b_2}{R^2} = 1 \tag{7}$$

Figure I will aid in clarifying the above discussion.

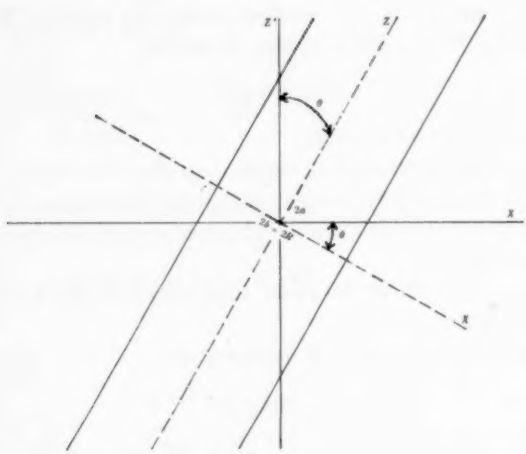

FIGURE I

RIGHT CIRCULAR CYLINDER WITH AXES ROTATED

Lemma. From the foregoing proof it can be stated that all planes intersecting a right circular cylinder form ellipses. The only limitation being as illustrated in (1).

It is interesting to note that the transformation of axes by rotation is usually reserved for simplification of equations. To obtain the Lemma it

was made use of for a different reason.

The important question remains, what relationship exists between the angle of the cutting plane, relative to the normal plane, and the eccentricity? This question will be answered in the following paragraphs.

It has been shown that a plane intersecting the axis of a right circular cylinder forms an ellipse, (Lemma).

The shape of the ellipse varies as a function of the angle of the cutting plane, relative to the plane normal to the axis of the surface. Obviously, the normal plane intersects the surface in a circle. The cutting plane must intersect the normal plane along a diameter of the circle. This is illustrated in Figure I.

Theorem. The eccentricity of an ellipse equals the sine of the angle between the intersecting plane and the plane normal to the axis of a right circular cylinder.

Given: Eccentricity $= e$

Angle between intersecting plane and normal plane $= \theta$

R = radius of right circular cylinder measured along the normal plane.

$2a$ = length of major axis of the ellipse

$2b$ = length of minor axis of the ellipse

$2c$ = distance between the foci of the ellipse

x = distance along X-axis

z = distance along Z-axis.

Proof:

$$\text{Sin } \theta = \frac{z}{a}, \quad \text{Sin}^2 \theta = \frac{z^2}{a^2}, \quad a^2 \text{Sin}^2 \theta = z^2 \tag{8}$$

$$\text{Cos } \theta = \frac{R}{a}, \quad \text{Cos}^2 \theta = \frac{R^2}{a^2}, \quad a^2 \text{Cos}^2 \theta = R^2 \tag{9}$$

$$\text{Eccentricity} = e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} \tag{10}$$

Due to the properties of projection, equation (10) can be rewritten as:

$$e = \frac{\sqrt{a^2-R^2}}{a} = \frac{\sqrt{a^2-a^2 \text{Cos}^2 \theta}}{a} \tag{11}$$

Equation (11) was modified by the use of equation (9). Equation (11) can be simplified by factoring, using a basic trigonometric identity, and taking the square root as follows:

$$e = \frac{\sqrt{a^2(1-\text{Cos}^2 \theta)}}{a} = \frac{\sqrt{a^2 \text{Sin}^2 \theta}}{a} = \text{Sin } \theta \tag{12}$$

Therefore, the eccentricity of an ellipse is equal to the sine of the angle between the plane normal to the axis and the cutting plane.

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(continued from page 132)
and therefore

$$\frac{S_{2n}}{2^n} = \sum_{0 \leqq 4r < 2n} \binom{2n-1}{4r+1} \frac{C_{4r+1}}{2^{2r}} \frac{S_{2n-4r-2}}{2^{n-2r-1}}. \tag{7}$$

Since, from (6),

$$C_{4r+1} = \frac{2^{4r+2}(1-2^{4r+2})B_{4r+2}}{4r+2}$$

and the denominator of B_{4r+2} contains the prime 2 to the first power, it follows that

$$C_{4r+1} = 2^{4r} C'_{4r+1},$$

where C_{4r+1} is odd. Hence, assuming the truth of (2) for all even integers less than $2n$, (7) yields

$$2^{-n} S_{2n} = -\sum_{0 \leqq 4r < 2n} \binom{2n-1}{4r+1} 2^{2r} C'_{4r+1} S'_{2n-4r-2}.$$

Hence (2) follows at once. Also since $C_1 = -1$, we get

$$S'_{2n} \equiv (-1)^{n-1} S_{2n-2} \pmod 4,$$

which implies (3).

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Duke University

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

MATHEMATICS IN THE ENGINEERING CURRICULUM

Robert E. Horton

An examination of course requirements in mathematics for engineering students in colleges throughout the United States today would reveal considerable variation in content, organization and amount. The courses listed by some colleges would differ very little from their requirements in the year 1900, while in other colleges the catalogue would look strange indeed to a student of the class of 1900. The interested reader is inclined to ask: "What are the fundamental differences between the programs offered by the various American colleges and universities for engineers? Why are changes in the curriculum occuring? What are the implications of these changes?"

In order to explore current thought on these questions a study of recent publications containing information pertinent to this subject was made. The findings have been organized to show current thinking on four important questions:

1. What are the major curriculum patterns in engineering mathematics in American colleges today?
2. What trends are evident in the changing mathematics curriculum for engineers?
3. What are the issues in the discussion of integrated courses as opposed to the traditional organization of mathematics?
4. What is the status of mathematics in the General Education phase of the engineering curriculum?

CURRICULUM PATTERNS

In an attempt to determine the major current curriculum patterns of engineering mathematics, catalogues from 50 American colleges were examined. Examination of these catalogues reveals several important facts.

Although there is a diversity among the colleges in their manner of organizing engineering mathematics, some elements remain which are common to all. Common elements include basic content; that is, all colleges include algebra, trigonometry, analytic geometry, differential and integral calculus in their required curriculum. Another common element is the placing of these studies in the first two years of the engineering course. The study of elementary differential equations is usually the most advanced mathematical concept required of engineering students. It should be noted that these common elements would have been found in the engineering curriculum of the latter part of the Nineteenth Century.

Differences in curricular treatment of engineering mathematics include methods of teaching the material and the inclusion of modern mathematical concepts. Differences in teaching methods revolve around the unified or integrated method versus the traditional compartmented treatment of the mathematics content. A fuller discussion of this matter will be made in a later part of this paper. However, it is sufficient to say that a large number of the colleges studied now have part or all of their engineering mathematics taught in an integrated or unified manner.

An enormous amount of entirely new mathematics has been developed in the last fifty years. Much of this mathematics has direct application to applied engineering problems. Two urgent questions face the curriculum planners of today. Which ones, and how many of the new mathematical techniques should be included in the engineering curriculum? Secondly, how can these new subjects be included in the already overcrowded engineering program? In a report of the Pasadena Conference on Mathematics in Engineering (22), sponsored jointly by the American Society of Engineering Education and the National Science Foundation, these questions were highlighted. The report says in part:

> The present rapid developments in the various engineering disciplines are bringing both the research and design engineer into an increasingly intimate contact with new and diverse fields of the mathematical sciences. Since this trend promises to increase steadily with future engineering advances, the role of mathematics in the engineering educational program should be under continual assessment.

The new subjects which are most urgently recommended for inclusion in the required curriculum are probability and statistics, modern algebras, modern electronic computation methods, and many topics in modern functional analysis. Bernard Ostle (16) says, "It is my belief that at least one basic course in statistics should be included in all undergraduate engineering curricula. Other statistics courses could be suggested as electives." Another point of view is indicated by M. E. Munroe (43):

> There are many conversations, committee meetings, etc., today about the modernization of the undergraduate calculus course; but all too often the attack on the problem falls short of being comprehensive. Calculus has been in cold storage for over fifty years now. These have

> been highly productive years in mathematics, and the result is that more changes are in order than most people would like to admit.
>
> The relevant branches of modern mathematics would seem to be real function theory and differential geometry....

A report of the Curriculum Subcommittee of the Pasadena Conference (22) makes the following recommendations for mathematics training beyond the calculus:

> The single course beyond calculus that we feel should be required of all engineering students is a one semester course in differential equations.... After differential equations, the next priority should be given to a one semester course in statistical analysis.... Our third recommendation is that the mathematical curriculum contain a two-year elective sequence in applied (mathematical) analysis for engineers.... include vector and tensor analysis, matrices, partial differential equations, special functions, boundary and initial value problems, complex variables, integral transforms, integral equations, and variational calculus.

One college program is presented here in some detail because it is typical of a new kind of emphasis in engineering training. It is a curriculum organized to produce research engineers, sometimes called design and development engineers. These men will practice the most advanced techniques of modern scientific research applied to engineering problems. Of course, a higher order of training than is given to most engineers is required. Cornell University is an example.

Cornell Engineering, Physics, Program

First Semester	Unified Analytics and Calculus
Second Semester	Unified Analytics and Calculus
Third Semester	Unified Analytics and Calculus
Fourth Semester	Differential Equations
Fifth Semester	Methods of Applied Mathematics
Sixth Semester	Methods of Applied Mathematics
Seventh Semester	Methods of Applied Mathematics
Eighth Semester	Methods of Applied Mathematics

A total of 24 semester units of mathematics is required and other mathematical specialities are elective. In writing about the demand for this type of engineering training, Professor Lloyd P. Smith of Cornell University (25) made the comment, "In spite of the fact that this course represents the most rigorous technical course in the university, the student demand for it has steadily increased."

One other aspect of curriculum patterns which is evident today is the increased emphasis on a year of graduate level study for engineers. Relative to the Engineering Physics program at Cornell, Professor Smith (25) says:

> It is at this point that one of the important objectives of the program is achieved, namely, that students completing the five-year course can go into industry immediately without further training and begin to make a significant contribution to the research or development program.

CURRICULUM TRENDS

Next let us turn our attention to the trends in the changing mathematics curriculum for engineering students. The first (obvious) trend is in the direction of requiring more formal mathematics in the engineering programs. In the Report of the Committee on Evaluation of Engineering Education published in 1955 (19) the statement was made:

> At the undergraduate level, competence in the theory and use of simple, ordinary differential equations and their applications to the solution of physical problems lies close to the boundary of minimum acceptability of mathematics in any satisfactory engineering curriculum.

Professor James H. Zant of Oklahoma A and M College (31) reinforces this idea. He says:

> It is plain to those working in the field of engineering education that the engineering student should study mathematics beyond the calculus. This should be partly for the purpose of securing an adequate mastery of the calculus itself and partly for added skills and knowledge not ordinarily included in the calculus courses.

It should be noted that some engineers consider the added emphasis on mathematics to apply only to certain highly technical branches of engineering. B. A. Whisler (29) states:

> Specifically, the current proposals for changes in engineering curricula take the form of requiring additional courses in higher mathematics and physics and other additions of this nature. Such changes are necessary for students who will later work in fields which require the specific background, but for the civil engineer such courses are of little significance.

The point of view of more liberal training in basic subjects is expressed by Richard B. Adler (1). In regard to the training of engineering leaders he said, "For such men it goes without saying that the maximum of science and mathematics should appear in the curriculum. Certainly four years of each would be reasonable."

A trend which seems to be in conflict with that of increasing the formal mathematics requirements is the practice of moving the responsibility for the teaching of mathematics from the mathematics department to the college of engineering. In noting this trend F. Virginia Rohde (23) writes:

> There seems to be a growing tendency in our engineering colleges to center the teaching of advanced mathematics at least within the engineering college itself rather than to have the mathematics department do it.... One reason frequently given is that there are already too many courses the student must take, and he does not have time for a separate mathematics course...., additional mathematics can be taught in the engineering courses as the need arises.

Professor C. O. Oakley expresses concern about this trend in an article entitled *Engineering for Mathematicians* (15). He states:

> The fact that there is a strong trend in departments of engineering to develop their own courses in mathematics rather than to continue the traditional practice of leaving the matter in the hands of the departments of mathematics is concrete evidence of dissatisfaction and is sufficient reason for deep concern, if not alarm.

It is natural to expect that pressure to include additional mathematics in an overcrowded curriculum would create pressure to eliminate some subject matter that is traditionally a part of the curriculum. Much thought is being given to the elimination of certain topics in algebra, much of the computational aspects of logarithms and of trigonometry and de-emphasis of the study of conics in analytic geometry.

A second plan to relieve the pressure of the mathematics curriculum is to place some of the traditional freshman mathematics down in the high school. By eliminating from high school courses such subjects as solid geometry and numerical solutions of triangles, some elements of analytic geometry and even beginning calculus might be mastered before the student enters college. H. Van Engen (27) points out, "Indeed, there is real reason to believe that a semester course in high school trigonometry cannot be justified in the light of the many demands for a knowledge of the more contemporary aspects of mathematics." In the same article he points out that numerous proposals are made to replace the subjects so removed from the high school by some elements of probability and statistics, the elements of analytic geometry and the elements of calculus. He feels that for schools which introduce algebra in the eighth grade, a full year of calculus in the senior year is possible.

In regard to the placement of students in the freshman college curriculum, Zant (31) advocates, "Start the student as far along in mathematics as he is capable of going. This may be in analytics or perhaps in some instances in the calculus." Differential placement of entering freshmen raises the problem of course credit. Some advocate giving courses credit for college level work taken in high school. Others permit only prerequisite credit for this work. There are no standard practices in this area as yet.

Zant (31) suggests that the college should ...

> Modernize the subject matter in mathematics. This can be done by giving a completely reorganized course, as suggested by a number of mathematicians, or it can be done in a more modest way by teaching the traditional subject matter from a somewhat different point of view. ... It also seems advisable that certain types of courses and subject matter be made available so that engineering students who recognize the need can take such offering. Three such areas are statistics, the basic principles of quality control and the computational mathematics....

Professor S. S. Wilks of Princeton University has this to say about the place of statistics in the college curriculum (49):

> One of the outstanding examples of an unfulfilled need for some training in probability and statistics is in engineering education. The statistical quality control movement which started during the war and which has now pretty well permeated American industry caught the engineering profession completely untrained in even the rudiments of probability and statistics. ... While the educational program in engineering schools is being improved to meet this need it is far from satisfactory. The main trouble is that special probability and statistics courses for

> engineers are upper class electives in most colleges and universities and are taken by a relatively small fraction of engineering students.... In my opinion they should be taught in the major freshman-sophomore sequence of mathematics courses.

Trends in the reorganization of the college mathematics courses to permit new material to be introduced also call for the elimination of certain topics from the usual freshman-sophomore courses. Rohde (23) says in this regard:

> Most of the time spent in analytic geometry on the conic sections (in other words most of the time spent in analytic geometry, in many cases) would seem to be wasted on the engineering student.... Probably the greatest objection to the teaching of calculus is the failure to emphasize the use of integral tables. As we all know, much time could be saved for more important things if the student were allowed to use tables in integrating functions such as $(3x+1)dx/(2x^2+5x+8)$. Again since the engineering courses cover thoroughly the subjects of moments, areas, volumes and the like, possibly some of this work should be omitted in the calculus and the time devoted to other topics.

Another specialized field of modern mathematics which is being brought into the engineering curriculum is that of computational mathematics. With the increase in use of electronic calculating equipment since World War II, the field of engineering is calling for more and more graduates highly trained in this subject. According to J. W. Tukey of Princeton University (46):

> ... mathematics departments should, in the writer's judgement, act as follows:
>
> (1) They should prepare to cooperate in the setting up of training for computation engineers under engineering auspices and,
>
> (2) They should prepare to teach the necessary mathematics including the mathematics of computation in the mathematics department.

Another trend in engineering education which seems to be gaining momentum is the increase in emphasis on graduate study for engineers. As John K. Wolfe, Manager of Advance Degree Recruiting for the General Electric Company puts it (30), "We feel that the college can hardly give a man in four years enough basic training in both science and engineering to make him a good nuclear engineer." The Summary of the Report on Evaluation of Engineering Education (19) has as its recommendation number 8, "The strengthening of graduate programs necessary to supply the needs of the profession." A survey conducted by Purdue University and reported by William K. Le Bold (11) found, "Industry indicated that during the next ten years a greater percentage of those with Master's and Doctor's degrees will be desired, but Bachelor's degrees will continue to constitute better than 80% of demand."

In pointing towards graduate study the engineering colleges are implementing special undergraduate programs which stress fundamental studies rather than training in the details of the profession. R. E. Purucker in a report on graduate study investigated by the Wisconsin University

Cooperating Committee (18) states:

> To implement the (graduate) program, the committee recommended the establishment of a four-year under-graduate course in 'Fundamental Science and Engineering'. ... Without thought of dictating specific subject matter, but still desiring to avoid having its recommendations misunderstood, the committee set up the following course of study: 28 credits in mathematics, 26 credits in chemistry, 67 credits in physical science and engineering and 24 credits in general subjects..

Such a program of undergraduate and graduate study is designed to produce a scientist-engineer quite different from the product of the usual four-year professional school. It is important to note the heavy emphasis on mathematics throughout the program. Earl P. Stevenson (26) says, "Meanwhile, mathematics has become the common language, and the first requirement of the scientist-engineer is that he be proficient in using mathematics as a real tool."

Along with the increase in the employment of scientist-engineers there has developed a need for a large number of semi-professional personnel trained in certain technical engineering specialties. Such personnel with about two years of collegiate level study or the equivalent of the Associate of Arts degree are needed to fill many of the more routine jobs of an engineering nature. The curriculum for this kind of semi-professional engineer or engineering technician also requires emphasis on certain branches of mathematics. In the Minutes of the Subcommittee on Engineering and Technical Institute Teachers of the University of California (56) we find information on this matter. In a summary of a survey of post high school mathematics necessary for technicians, a graph was drawn showing that a poll of 134 persons in 32 different companies revealed that the kind of mathematics used by most technicians lies in four areas. These are algebraic techniques, advanced trigonometric techniques, introductory calculus, and the use of the slide rule.

It is necessary to report one more trend in the mathematics curriculum for engineers. This is the trend towards more unification or integration of subject matter in the engineering program. This trend appears in two forms. One is the integration of subject matter within the mathematics courses. The other is seen in the unification of mathematics instruction with other in science and engineering subjects. The integration of subject matter will be discussed at some length in the next section of this article.

To summarize briefly the major trends apparent in the engineering mathematics curriculum, we can list the following:

1. The increase in the total number of credits in mathematics required of most engineering programs.

2. The move to have the mathematics courses for engineers taught in the engineering departments by engineering professors.

3. The introduction of modern fields of mathematics into the curriculum.

The subjects most frequently added are probability and statistics, Boolean algebra, modern functional analysis, and modern computational mathematics.

4. A move is evident to study more advanced mathematics earlier in the course. It even sees calculus and analytic geometry as senior year high school subjects.

5. Heavier emphasis is developing in the graduate training of engineers.

6. Finally a strong trend is noticed in the direction of unifying subject matter both within the mathematics courses and throughout the technical aspects of the engineering program.

UNIFIED COURSES

For our purpose I shall take as the definition of an integrated or unified course one in which subject matter, that usually is treated in separate courses, is brought together and taught as a related whole. Two kinds of unification of material are apparent in studying modern mathematics curricula. One is an internal unification of various mathematical subjects in one course. The other is a unification of mathematical and essentially non-mathematical subjects in one course. For convenience I shall call the first of these the inter-mathematics type of unification and the second the mathematics-science-engineering type of unification.

In speaking of the inter-mathematics type of unification, James H. Zant (31) says:

> Integrate the subject matter of mathematics to such an extent that the students will get the idea that all mathematics is built on a logical basis and involves undefined terms, definitions, postulates, and theorems proved from these, using the accepted rules of logic.

The pressures to unify mathematics courses come from many sources in the modern engineering curriculum. S. E. Urner (53) points out two of these:

> The search for a unified approach to Freshman Mathematics appears to stem principally from two sources:
> (1) The philosophical or theoretical, recognition that the cultural and disciplinary values of mathematics may be enhanced by such an approach;
> (2) The practical need for early introduction of the calculus in the training of prospective engineers and scientists.

Other proponents of inter-mathematics unification point out the possibility of introducing more modern mathematical concepts into the already crowded curriculum. The inclusion of these concepts early in the program is stressed. Andre Weil (48) says:

> For instance the idea of the function, the process of differentiation and integration, should appear at an early stage because of their enormous importance both for the theory and for most ordinary practice. Because of its importance, numerical calculation, and all the devices, connected with it, would seem to deserve a far more prominent place in elementary (college) teaching than they receive at present.

In criticism of the teaching of mathematics in separate compartmented

courses, Robert B. Davis (4) makes the point:

> Yet we often divide the curriculum into vertical segments with far too much separation, and require an unrealistic degree of mastery at each step before allowing the student access to the next higher step in the ladder.

Davis states further,

> We also divide the curriculum into horizontal segments, and teach—for one example—integration as a dull (and difficult) mathematical technique, where the practical-minded student finds little to seize hold of. We teach differential equations as a separate subject. The study of beam bending is again separate. Yet these three areas can be effectively combined, in a certain sense.

In considering the assimilation of new scientific knowledge into a unified curriculum, the Committee on Evaluation of Engineering Education had this to say in its Report (19):

> This translation of new scientific developments into engineering practice will be facilitated by emphasizing unity in scientific subject matter. ... When a student understands these generalizations, he has gained a concept of systematic orderliness in many fields of science and engineering; he is therefore able to approach the solution of problems in widely diverse fields, using the same analytical methods. This unification of methods of analysis can be accomplished to a considerable degree without reaching beyond undergraduate mathematical levels. It can be accomplished to a much greater degree by utilizing advanced mathematical concepts.

A summary of arguments for and against integrated mathematics courses is made by James H. Zant (31):

> Among the advantages (of integrated courses) are the natural ones of seeing related topics discussed together and thus being able to realize that the same rules and theorems usually apply all through the study of mathematics. It is also possible to get a truer picture of a concept when the student has the advantage of viewing it as it appears in several areas.
>
> ...
>
> Some of the disadvantages are that the books and courses are shorter than traditional ones and may therefore be more difficult. Also, and for the same reasons, less drill material is included, so that students with weaker backgrounds may not acquire needed skills.
>
> Another disadvantage is the inflexibility of the course. When a student has begun a course in mathematical analysis that covers three or four semesters, he may be at a distinct disadvantage if he transfers to another institution having other kinds of courses or even using a different text book. Students transferring into college may have the same sort of difficulty. This is a very real problem in many large midwestern state institutions.

The most common type of inter-mathematical unification is the combining of elements of algebra, analytic geometry, differential and integral calculus and trigonometry into one course, often called unified freshman mathematics or beginning mathematical analysis. Examples of this type are found in the curriculum organizations of Brown University, University of California at Los Angeles, and Ohio University. Text books written to serve in this type of course include, among many others: *An Introduction to Mathematical Analysis* by F. L. Griffin published in 1936; *Introductory*

College Mathematics by W. E. Milne and D. R. Davis published in 1935; and *Elements of Mathematical Analysis* by S. E. Urner and W. B. Orange published in 1950. Many more books slanted toward this approach are now on the market.

Another pattern of unified freshman mathematics introduces not only unification of the usual topics of analysis but also brings in other topics of modern mathematics. Haverford College conducts one such course. Several recent texts have been written to fill the needs of this type of program. One is the text *Principles of Mathematics* by C. B. Allendoerfer and C. O. Oakley published in 1955 (55). In their preface the authors say:

> This book has been written with the conviction that large parts of the standard undergraduate curriculum in mathematics are obsolete, and that it is high time that our courses take due advantage of the remarkable advances that have been made in mathematics during the past century.

Such topics as Logic, Groups, Fields, Sets and Boolean Algebra, and Statistics and Probability are included. It is impossible to mention all the new texts that are organized around some unifying element or mathematical idea. It is sufficient to say that there are enough to indicate that unified mathematical courses are rapidly growing in favor with college faculties.

The mathematics-science-engineering type of unification is typified by a program conducted at Cornell University. Speaking of this course Dwight F. Gunder (9) remarked:

> The subject matter chosen for integration was: mathematics through elementary differential equations, most of the topics from a standard college physics course, statics, dynamics and strength of materials, a first course in materials testing and a first course in basic electrical engineering.
>
> It was proposed to replace these by: a single course in engineering fundamentals, a one-year course in mathematics in the third or fourth year of the curriculum, and a one year course in physics in the third or fourth year of the curriculum.

In another context V. L. Parsegian (17) stated:

> A science package and its objectives have been proposed in which chemistry, physics and mathematics play a more integrated and effective role in strengthening the preparation of engineers for their new, more responsible engineering functions in a more complex engineering world.

Morris Kline (40) considers the unity of ideas when he states:

> My philosophy contains three principles. The first of these states that knowledge is a whole and that mathematics is part of that whole. However the whole is not the sum of its parts. The present procedure is to teach mathematics as a subject unto itself and somehow expect the student who takes only one year of the subject to see its importance and significance for the general body of knowledge.... It follows from this principle that mathematics must be taught in the context of human knowledge and culture.

John W. Cell (3) points out that the unification of mathematics with other subjects has advantages for mathematics learning. He says:

> In colleges where calculus is used in the instruction in physics and where the teaching of mathematics and physics is somewhat correlated, there is an automatic motivation for the student in his study of mathematics. This motivation is necessary if students are to learn elementary mathematics and not merely earn course credits.

To summarize the discussion of integrated courses versus traditional courses, the following points are made in favor of integration or unification:

1. Integrated courses can emphasize the logical unity of mathematics as well as relate it to all knowledge.
2. Integrated courses are needed in order to introduce calculus and other advanced ideas into the freshman mathematics curriculum at an early stage.
3. Integration is necessary to create room for the inclusion of new kinds of mathematics in the curriculum.
4. Perhaps the strongest argument in favor of unified courses is their capacity to increase the motivation for careful mathematics study by students.

Arguments against integrated courses include:

1. The texts and courses in use tend to be more difficult and have less drill material for the weaker students.
2. Unified courses tend to complicate the problem of transfer of students from one college to another.

GENERAL EDUCATION

Finally some mention should be made regarding the place of mathematics in the General Education part of the engineering curriculum. An examination of current literature reveals that engineering faculties regard mathematics almost exclusively as a technical tool. In the report on General Education in Engineering (20) which occupied the entire April 1956 issue of the Journal of Engineering Education, mathematics was not considered as general education. To the engineer it seems that General Education is to be considered as synonymous with Humanistic-Social subjects. A few references can be found that consider mathematics in the general education of students such as Morris Kline (40), Report of the Special Committee on College Mathematics (44), Hartley Rogers Jr. (45), and Robert E. Horton (50). However, these are mostly concerned with the general education of students in the non-scientific curricula. Is it reasonable to expect that our engineers will be less familiar with the cultural, historical, and aesthetic aspects of mathematics than are other students? Or is it assumed that these concepts are developed in the usual engineering mathematics courses?

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MISCELLANEOUS NOTES

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THE MODERN MATHEMATICAL APPROACH TO LR²H

R. F. Rinehart

A concomitant of the burgeoning of modern mathematics has been the increased use of ingenious symbolism to reduce the writing involved in the communication of mathematical ideas. It seems regrettable that contemporary composers of literature have not seen fit to streamline their communication methods similarly. Through ignorance, or design, they remain blissfully aloof to the opportunities afforded to incorporate greater precision in, and to inject a new austere beauty into, their works. Consider, for instance, how much more satisfying and appealing the following well-known classic becomes, when clothed with the new notational raiment.

$\exists\ t_0 \ni$ at $t = t_0 \exists$*a small girl, denoted by Little Red Riding Hood (notation: LR^2H). LR^2H left her home, taken as origin, to pay a visit to $f(f(LR^2H))$, (hereafter denoted by G), where f is the Murrow mapping** (daughter) $\to$ (mother). Her purpose in the visit was the construction of the set $G \cup S$, S being a certain finite subset of the set $C \cup K \cup Q$, where $C = \{\alpha : \alpha = \text{candy}\}$, $K = \{\beta : \beta = \text{cookie}\}$, and $Q = \{\gamma : \gamma = \text{other goodie}\}$.

Now $G \subset F$, a forest, and the set $W_1 = \{w : w \text{ alive}; w \in W\}$, where $W = \{w : w = \text{wolf} \in F\}$, was not empty. At $t = t_1 > t_0$, LR^2H met $w_1 \in W_1$, who inquired about the zeros for $t > t_1$, of $\dot{P}(LR^2H)$, $P(X)$ denoting the position vector of the argument X, and the dot denoting, as usual, the time derivative. On learning that the first zero was at $X = G$, w_1 embarked on a minimal variational fixed-end-point path with termini $P(LR^2H)$ and $P(G)$. Arriving at the cottage of G, (at $t = t_2$) he effected the transformation $w_1 \cup G = w_1$, and donning the nightgear of G ensconced himself in the latter's bed.

At $t = t_3 > t_2$, the condition $|P(LR^2H) - P(w_1)| < 8$ feet, was realized. To describe adequately the ensuing conversation it is necessary to introduce some notation. Let $F_1(x)$, $F_2(x)$ and $F_3(x)$ denote, respectively, the

*Note the gain in precision over the time honored "Once upon a time, there was ..."
**Also known as the *person-to-person* mapping.

ear length, eye brightness, and tooth sharpness of x. Let $H_1(y)$, $H_2(y)$ and $H_3(y)$ denote, respectively, the efficiency of w_1's acquisition, from y, of energy of the following types: (1) acoustical, (2) visible electromagnetic, and (3) nutritional. The following conversation then took place, where $E(z)$ denotes, as usual, the expected value of the variate z:

$$\left.\begin{array}{l} LR^2H: \text{"}F_i(\text{you}) \gg E(F_i(x)),\ G!\text{"*} \\ w_1: \text{"}F_i(\text{me}) \text{ maximizes } H_i(\text{you}), \text{ dear } \epsilon \text{ me."} \end{array}\right\} i = 1,2,3$$

With the last remark for $i = 3$, w_1 leaped out at LR^2H.

But, at that very $t = t_4$, a woodcutter who had random-walked by and overheard the conversation, burst into the cottage and axeally transformed w_1 into $\bar{w}_1 \epsilon \bar{W}_1$, the complement of W_1 in W. Now fortunately, $\bar{w}_1$ was separable, with G as one component, and the universe of discourse $\exists$ happily $\forall\ t > t_4$.

*The ! here is an archaic symbol used in literature, and does not denote the factorial, or Grandma, function.

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AN INFINITE SET OF FORMULAS CONNECTING BINOMIAL COEFFICIENTS

R. F. Graesser

Blaise Pascal was probably the first mathematician to investigate the properties of the binomial coefficients. For this reason the arithmetic triangle is often called Pascal's triangle. During the nineteenth century a large number of formulas connecting the binomial coefficients were published. The following simple procedure gives an infinite set of such formulas.

If n and i are integers greater than zero, then

$$(1+x+x^2+\cdots+x^n)^i = \left(\frac{1-x^{n+1}}{1-x}\right)^i = (1-x^{n+1})^i(1-x)^{-i} \tag{1}$$

$$= \sum_{r=0}^{i} (-1)^r \binom{i}{r} x^{r(n+1)} \sum_{k=0}^{\infty} \binom{i+k-1}{k} x^k \tag{2}$$

$$= \sum_{r=0}^{i} \sum_{k=0}^{\infty} (-1)^r \binom{i}{r} \binom{i+k-1}{k} x^{k+r(n+1)}$$

We want the coefficient of x to the power $ni+t$ in this expansion, where t is a non-negative integer. Hence

$$k+r(n+1) = ni+t,$$

and

$$k = ni+t-r(n+1)$$

so that the desired coefficient is

$$\sum_{r=0} (-1)^r \binom{i}{r} \binom{i+ni+t-r(n+1)-1}{ni+t-r(n+1)} = \sum_{r=0} (-1)^r \binom{i}{r} \binom{i+ni+t-r(n+1)-1}{i-1}$$

$$= \sum_{r=0} (-1)^r \binom{i}{r} \binom{(i-r)(n+1)+t-1}{i-1}, \tag{3}$$

where the upper limit of r in these summations is to be determined.

From the first factor in the right member of (2), it is seen that $r(n+1)$ can not exceed $ni+t$. Since r is a non-negative integer, it can not exceed the largest integer in $(ni+t)\div(n+1)$, which may be designated $[\frac{ni+t}{n+1}]$. From the upper limit of the summation in the first factor of the second member of (2), r is less than or equal to i. Hence the upper limit of r in (3) is the lesser of the two integers i or $[\frac{ni+t}{n+1}]$, which we might call R. It might be noted that these restrictions also avoid impossible forms for the symbols $\binom{i}{r}$ and $\binom{(i-r)(n+1)+t-1}{i-1}$.

Because the highest power of x in both members of (1) must be ni, we have the following infinite set of formulas connecting binomial coefficients:

$$\sum_{r=0}^{R}(-1)^r\binom{i}{r}\binom{(i-r)(n+1)+t-1}{i-1} \begin{array}{l} = 1 \text{ if } t=0, \\ = 0 \text{ if } t=1,2,3,4,\cdots. \end{array} \tag{4}$$

As a numerical illustration of (4), we might take $n=5$, $i=8$, and $t=3$. Then $[\frac{ni+t}{n+1}] = [\frac{43}{6}] = 7$, which is the value of R so that we have

$$\sum_{r=0}^{7}(-1)^r\binom{8}{r}\binom{(8-r)6+2}{7} = \sum_{r=0}^{7}(-1)^r\binom{8}{r}\binom{50-6r}{7}$$

$$= \binom{8}{0}\binom{50}{7}-\binom{8}{1}\binom{44}{7}+\binom{8}{2}\binom{38}{7}-\binom{8}{3}\binom{32}{7}+\binom{8}{4}\binom{26}{7}-\binom{8}{5}\binom{20}{7}+\binom{8}{6}\binom{14}{7}-\binom{8}{7}\binom{8}{7}$$

$$= 1(99884400)-8(38320568)+28(12620256)-56(3365856)$$
$$+70(657800)-56(77520)+28(3432)-8(8)$$
$$= 0.$$

University of Arizona

MATRIX MANIPULATOR

In computation involving matrices it is frequently necessary to interchange or rearrange rows or columns of a matrix. If the work is being done longhand or with a desk calculator, it is desirable to be able to perform the rearrangements without having to erase or rewrite numbers. A simple mechanical matrix manipulator which achieves this result has been designed by D. M. Mesner of the National Bureau of Standards. The possibility of designing a desk-size electronic instrument to perform the same operations is being considered.

The matrix manipulator was devised for use in the Bureau's Statistical Engineering Laboratory for calculations with incidence matrices, i.e., matrices whose elements are all 0's or 1's. Matrices of this kind arise in many branches of science and pure mathematics, as well as in statistical analysis. The solution of systems of linear equations, perhaps the best known application of matrices, should also be facilitated by the manipulator.

The device consists of a set of small plastic blocks which can be arranged in a rectangular array to serve as cells of a matrix. The upper faces are sanded to provide writing surfaces for pencil or pen. The blocks are drilled with 2 horizontal holes each through which a metal rod can be passed, so that an entire row or column can be lifted out and replaced where desired.

If arithmetical operations result in new numbers in the matrix, the old numbers can be erased and the new ones put in their place about as easily as on paper. The blocks could also be replaced by others containing the new numbers. When working with incidence matrices, where only 0's and 1's occur, the matrix pattern can be more easily distinguished if the 1's are represented by blocks with a black top while the 0's are left white.

Comparing Networks

A good illustration of a problem which can be solved by rearranging rows and columns of a matrix is that of determining whether or not 2 networks have the same structure. An example is the diagrams of 2 networks each containing 8 boxes connected by lines. If the lines are interpreted as the appropriate form of connection or communication, then such networks might represent the connections in an electrical circuit, the flow of information between components in a computer, the channels of communication between departments of an organization, or the structural formula of a molecule, to mention a few of the possibilities. Whatever the

interpretation of the diagram, any network can be represented in concise mathematical form by an incidence matrix with rows and columns labeled to correspond with the boxes. In such a matrix, the entry in position x of row y is set equal to 1 if boxes x and y are directly connected, or it is set equal to 0 if the same boxes are not so connected. The incidence matrices for networks I and II are given in Table 1.

Table 1. Incidence matrices.

Network I

	1	2	3	4	5	6	7	8
1	0	1	0	0	1	1	0	0
2	1	0	1	0	0	0	1	0
3	0	1	0	1	0	0	0	1
4	0	0	1	0	0	1	0	1
5	1	0	0	0	0	1	1	0
6	1	0	0	1	1	0	1	0
7	0	1	0	0	1	1	0	1
8	0	0	1	1	0	0	1	0

Network II

	A	B	C	D	E	F	G	H
A	0	1	1	0	1	0	0	0
B	1	0	0	1	0	1	0	1
C	1	0	0	1	0	0	0	1
D	0	1	1	0	0	0	0	1
E	1	0	0	0	0	1	1	0
F	0	1	0	0	1	0	1	0
G	0	0	0	0	1	1	0	1
H	0	1	1	1	0	0	1	0

Suppose it is desired to know whether the 2 networks are as different as they appear, or whether they really have the same structure and differ only in the way the diagrams have been arranged on the paper. Put in another way, the problem is to find whether the boxes of network II can be relabeled with the numbers 1 to 8 in such a way that the pairs of boxes directly connected by lines in network II have exactly the same numbers as the connected pairs in network I – and vice versa.

However, renumbering the boxes of a network in a different order has exactly the same effect as rearranging the rows and columns of the incidence matrix in the corresponding order. Thus the networks are identical if we can find a rearrangement (permutation) of the rows and columns of the second matrix which will give it exactly the same pattern of 1's and 0's as the first matrix.

The matrix manipulator makes it easy to try different arrangements. In working with an 8×8 matrix, an extra row and an extra column are added for carrying the numbers or letters that index the rows and the columns; the index row and column furnish a record of the rearrangement that has been carried out and provide a check that the same rearrangement has been used for rows as for columns.

In practice, it is convenient to set up both matrices on blocks and try to rearrange each of them into a common form. In the example of networks I and II, it is helpful to begin by placing the 2 rows that contain four 1's in leading position, then doing the same for the columns. A few trials are sufficient to find rearrangements that make the matrices identical, showing

that the 2 networks do indeed have the same structure. When the matrix for network I is left unchanged, the rearrangement found for network II and its incidence matrix is : *C A E G D H B F*

US-COMM-NBS-DC

ON TWO FAMOUS INEQUALITIES

L. H. Lange

In December of 1958 the Commission on Mathematics of the College Entrance Examination Board, established in 1955, published its report. The report contains a nine-point program designed to meet the needs of contemporary college-capable students of mathematics. One of these points concerns "treatment of inequalities along with equations" and it is a point well taken.[1] In the present note we take this point to heart. We discuss briefly several elementary but worthwhile inequality problems and, secondly, we call attention to a rewarding source of material on inequality problems and their solution.

We begin with the simplest special case of the most famous theorem belonging to the subject of inequalities, the *theorem of the arithmetic and geometric means*.[2] If a and b are (positive) real numbers, then $0 \leqq (a-b)^2$, with equality taking place if and only if $a = b$. Equivalently, then $2ab \leqq a^2 + b^2$. Letting $x = a^2$, $y = b^2$, we see immediately that what we are really dealing with here is the following relation between the geometric and arithmetic means of the positive numbers x and y: $\sqrt{xy} \leq \frac{x+y}{2}$, with equality occurring if and only if $x = y$.

In its various forms this inequality appears in many contexts, of course. Here is an application which apparently does not occur in the texts. We consider the problem of determining the minimum initial speed, v_0, required to drive a golf ball 300 yards – where we assume the fairway to be level, the atmosphere non-meddling, and a terminal roll of 25 yards. Letting the initial velocity vector of the golf ball have a horizontal component of magnitude $a > 0$ and a vertical component of magnitude $b > 0$, we have $v_0^2 = a^2 + b^2$. In all cases involving this type of projectile flight, the *range* of the projectile is given by $\frac{2ab}{g}$, where g is the gravitational constant. (This follows from the fact that the parametric equations for the projectile path are $x = at$, $y = -\frac{gt^2}{2} + bt$, and the equation $y = 0$ has the

positive root $\frac{2b}{g}$. Here x, y, and t are to be assigned their standard meanings.) In our problem the range is a given number, 275 yards. Hence, in particular, the product $2ab$ is a fixed number. Then, since $2ab \leqq a^2 + b^2$, the theorem of the means tells us that the minimum of $a^2 + b^2$ occurs if and only if $a = b$. It follows that we may very easily now compute the minimum, v_0, since we may therefore write

$$275 \text{ yards} = \frac{2ab}{g} = \frac{a^2+b^2}{g} = \frac{v_0^2}{g}.$$

We may give the approximate answer $v_0 = 160$ feet per second.

We now concern ourselves briefly with another famous inequality, the *Cauchy inequality*. Here too we consider only a special case, though the proof in this case is easily modified to dispose of the general case as well.[3]

Consider the expression

$$T = (a_1 - tb_1)^2 + (a_2 - tb_2)^2 + (a_3 - tb_3)^2,$$

where t is a real number and $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$. Unless there exists a constant k such that $a_i = kb_i$ for $i = 1, 2, 3$, we have $T > 0$ for all t. Hence, if the a's and b's are *not* proportional, the following quadratic equation in t,

$$(t^2)\sum b_i^2 + (t)(-2\sum a_i b_i) + \sum a_i^2 = 0,$$

must have imaginery roots. Since this implies that the discriminant is negative, we have

$$(-2\sum a_i b_i)^2 - 4(\sum b_i^2)(\sum a_i^2) < 0.$$

Thus we can arrive at Cauchy's inequality:

$$(\sum a_i b_i)^2 \leqq \sum a_i^2 \sum b_i^2,$$

with equality occurring if and only if the a's and b's are proportional.

We now take a problem from a book which is rich in inequality problems. It is G. Pólya's *Induction and Analogy in Mathematics*,[4] and we discuss part (2) of problem 61, page 141, "Given E, the sum of the lengths of the twelve edges of a box, find the maximum (1) of its volume V, (2) of

its surface S." (Pólya has earlier replaced the lengthy "rectangular parallelepiped" by "box".)[5]

Letting x, y, z be the lengths of the three edges drawn from the same vertex of the box, we have $E = 4(x+y+z)$ and we can state our problem this way: *Maximize* the expression $S = 2(xy+yz+zx)$ under the side condition $x+y+z = \frac{E}{4}$ = a positive constant. Now, since $(\frac{E}{4})^2 = x^2+y^2+z^2+S$, our problem is to *minimize* $S' = x^2+y^2+z^2$ under the same side condition. If we dwell on the fact that Cauchy's inequality contains such a sum of squares, we can make the pleasant discovery that it solves our problem if we set $a_1 = x$, $a_2 = y$, $a_3 = z$; $b_1 = b_2 = b_3 = 1$. Then

$$(x\cdot 1+y\cdot 1+z\cdot 1)^2 \leqq (x^2+y^2+z^2)(1^2+1^2+1^2);$$

i.e.,

$$\text{(I)} \qquad (x+y+z)^2 \leqq 3(x^2+y^2+z^2).$$

Thus

$$(\frac{E}{4})^2 \leqq 3S',$$

where S' is a minimum if and only if $x = y = z = \frac{E}{12}$. Our box with minimum surface area is a cube.

To be sure, the problem can be solved without an appeal to Cauchy's inequality, either as Pólya does it,[6] or by simply establishing our inequality (I) directly by adding the four inequalities:

$$2xy \leqq x^2+y^2$$

$$2yz \leqq y^2+z^2$$

$$2zx \leqq z^2+x^2$$

$$x^2+y^2+z^2 \leqq x^2+y^2+z^2.$$

FOOTNOTES

1. See point number 5 of the list given on pages 773-774 of the *American Mathematical Monthly* for December, 1958.

2. For several proofs of the *general* statement of this theorem see G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1952, pp. 16-21.

3. See *Inequalities*, p. 16.

4. This is volume I of his two volume set called *Mathematics and Plausible Reasoning* and published by Princeton in 1954. After each chapter in these volumes, Professor Pólya places a sequence of problems. In the spirit of Pólya

and Szegö, *Aufgaben und Lehrsätze aus der Analysis* I, II (Berlin, 1925), he later gives solutions to these problems.

5. Part (1), which is perhaps easier than part (2), we leave for the reader. Pólya's solution takes but one line on page 257 of that volume.

6. See pages 257-258 of *Induction and Analogy in Mathematics*.

University of Notre Dame

Mass education is apt to be like that other mass leavening process, the manufacture of store bread. The product comes out on time, but the flavor is missing.

Anonymous

GROUP THEORY AND COLORS

A. R. Amir-Moéz

1. DEFINITION: Let us call a set of objects S, and denote the objects in S by a, b, c, $\cdots$. A single-valued binary operation 'o' on S is a rule which assigns to each pair of objects a and b of S a unique object $p = aob$ which may or may not be in S. If for any two objects a and b of S, $p = aob$ is also an object in S, then we say S is closed under 'o'.

2. EXAMPLE: Suppose S is the set of all the negative integers and the binary operation is multiplication. It is clear that S is not closed under multiplication.

If for the same S we take addition as the binary operation, then S is closed under addition.

3. DEFINITION: A set S of objects is said to be commutative under a single-valued binary operation 'o' when for any two objects a and b of S we have $aob = boa$.

S is called associative under 'o' if for any three objects a, b, c, of S we have $ao(boc) = (aob)oc$.

4. EXAMPLE: Clearly the set of all the positive integers is commutative and associative under multiplication.

5. DEFINITION: [2] A set of objects G is called a commutative group under a single-valued operation 'o' if:

I G is closed under 'o'

II G is commutative and associative under 'o'

III There is an object e in G, called the identity such that for any object a of G we have $eoa = a$.

IV For any object a of G there is another object a' in G, called the inverse of a, such that $a'oa = e$.

6. DEFINITION: We borrow the following from the algebra of sets. If $X = \{x, y, \cdots\}$ is a set whose elements are x, y, $\cdots$, and $A = \{a, b, \cdots\}$ is another set whose elements are a, b, $\cdots$, then $X + A$ means the set which contains all the elements of X and all the elements of A. That is $X + A = \{x, y, \cdots, a, b, \cdots\}$.

7. COLORS AS A GROUP: Let $X = \{x, y, \cdots\}$, where x, y, $\cdots$ are dominant and complementary wavelengths [1]. Let $A = \{w\}$ be the set whose only element is w, white. Then

$$G = X + A = \{x, y, \cdots\} + \{w\}$$

is a commutative group under the operation of combining the elements of G. Here w is the identity and the inverse is the complement. The reader is requested to examine G for the properties given in 5.

REFERENCES

[1] T. Y. CROWELL, "THE SCIENCE OF COLOR", pp 236-246, N. Y. 1953

[2] C. C. MAC DUFFEE, "AN INTRODUCTION TO ABSTRACT ALGEBRA", pp 47, 48, N. Y. 1940

Queens College

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

362. *Proposed by David L. Silverman, Greenbelt, Maryland.*

Not having colored ink with which to dot the foreheads of his three apprentices, the wizard wrote numbers on their foreheads instead and told the apprentices that each had been given a prime number, the three of which formed the sides of a triangle with prime perimeter. The apprentice who deduced his number first was to be the wizard's successor. Apprentice A was given a 5 and B a 7. After a few minutes of silence C was able to deduce his number. What was it?

363. *Proposed by Brian Brady, Richmond, N.S.W., Australia.*

OAB is a line and P is a point which moves along a line through O which makes an angle α with OAB. If $OA = a$, $AB = b$ find the maximum value of angle APB.

364. *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

Find

$$\lim_{x\to\infty} (x \tan 1/x)^{3x^2}$$

365. *Proposed by Robert E. Shafer, University of California Radiation Laboratory.*

Find the cartesian coordinates of all points in an infinite hexagonal lattice, given the circumscribed circle radius of a hexagonal cell as unit length.

366. *Proposed by George Bergman, Stuyvesant High School, New York.*

Show that an infinite number of complex numbers z satisfy the equation $e^z = z$.

367. *Proposed by J.B.Love, Eastern Baptist College, Pennsylvania.*

Let $\phi(x)$ and $\psi(x)$ be monotone nondecreasing functions as $x\to\infty$ and both be positive for $0<x<\infty$. Let $f(x)$ be defined for $x>0$ with $f'(x)$ and $f''(x)$ existing such that $|f(x)|<\phi(x)$ and $|f''(x)|<\psi(x)$. Show that

$$|f'(x)| < 2[\phi(x)\cdot\psi(x)]^{1/2}$$

368. *Proposed by Warren G. Preble, Chalmette, Louisiana.*

Find the continuous segment of a conic section which provides the best least square fit to the five points (0, 0), (1, 0), (−1, 0), (0, 1), (0, −1).

SOLUTIONS

Late Solutions

336. *C.W.Trigg, Los Angeles City College.*

Erratum. In problem 350, page 47 the denominator of the second fraction in the parentheses should read $-\dfrac{2}{e^{1/x}+1}$

A Star Minus Rats

341. [May 1958] *Proposed by James H. Means, Huston-Tillotson College.*

In the subtraction $STAR-RATS = TRSA$ the minuend, subtrahend and difference are composed of the same four digits. Replace the letters with numerical digits. Is the solution unique?

Solution by C.W.Trigg, Los Angeles City College. Method **I.** Each of the four-digit integers is composed of the same digits, so $TRSA$ and therefore $T+R+S+A = 0 \pmod 9$. Furthermore, from the subtraction $R+T = S$ or $S-1$. Also, $A+S = R+10$.

From the fourteen sets of digits which meet the first condition, fourteen arrangements can be made which meet the second condition. These are (R, T, S, A) = (2, 6, 9, 1), (1, 7, 8, 2), (1, 3, 5, 9), (3, 5, 9, 1), *(1, 6, 8, 3), (1, 4, 5, 8), (1, 4, 6, 7), *(1, 6, 7, 4), *(3, 2, 5, 8), (2, 5, 8, 3), (3, 5, 8, 2), *(3, 2, 6, 7), *(2, 4, 7, 5), and (2, 5, 7, 4). Of these, only the five starred ones meet the third condition. Hence, the unique solution is 7641 − 1467 = 6174 in the decimal scale of notation.

Proceeding in like manner in the scale of 9, we find the unique solution 5173 − 3715 = 1357. There are no solutions in the scales of 5, 6, 7, 8, 11, or 12.

Method **II.** $S>R$, so $A+S = R+10$, and the subtraction can be interpreted as a set of simultaneous equations in four possible ways:

1) $A+S=R+10,\ S+T+1=A,\ R+A=T,\ R+T=S.$

This set leads to the impossible, $S+R+1=0$.

2) $A+S=R+10,\ S+T+1=A,\ R+A=T+10,\ R+T+1=S.$

This leads to $R=17/5$.

3) $A+S=R+10,\ S+T+1=A+10,\ A+R+1=T+10,\ R+T+1=S.$

This leads to $R=34/5$.

4) $A+S=R+10,\ S+T+1=A+10,\ A+R+1=T,\ R+T=S.$

This leads to the unique solution:

$S=7,\ T=6,\ A=4,\ R=1.$

Or, $7641-1467=6174$.

Also solved by William E.F. Appuhn, St. John's University, New York; George Bergman, Stuyvesant High School, New York; D.A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Monte Dernham, San Francisco, California; Helen Furtado, Salve Regina College, Rhode Island; J.M. Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; H.M. Gehman, University of Buffalo; Edgar Karst, Endicott, New York; John Q. Taylor King, Huston-Tillotson College, Texas; Herbert R. Leifer, Pittsburgh, Pennsylvania; Erich Michalup, Caracas, Venezuela; Charles F. Osgood, Haverford College Pennsylvania; Michael J. Pascual, Burbank, California; Charles F. Pinzka, University of Cincinnati; Lawrence A. Ringenberg, Eastern Illinois University; William M. Sanders, Mississippi Southern College; Frank W. Saunders, Coker College, South Carolina; A. Spinak, University of California at Los Angeles; Sister M. Stephanie, Georgian Court College, New Jersey; Walter R. Talbot, Jefferson City, Missouri; Dale Woods, Idaho State College; and the proposer.

A Sub-factorial Congruence

324. [May 1958] *Proposed by C.W. Trigg, Los Angeles City College.*

Show that $!n \equiv n! \pmod{n-1}$, where $!n$ is sub-factorial n.

Solution by Michael J. Pascual, Burbank, California.

$$!n = n!\left[\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right],$$

so that

$$n!-!n = n!\left[1-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^n}{(n-1)!}+\frac{(-1)^{n+1}}{n!}\right]$$

$$=(n-1)n(n-2)!\left[1-\frac{1}{2!}+\cdots+\frac{(-1)^n}{(n-1)!}+\frac{(-1)^{n+1}}{n!}\right]$$

$$= (n-1)\left[m + \frac{(-1)^n n}{(n-1)} + \frac{(-1)^{n+1}}{n-1} \right]$$

$$= (n-1)\,[m + (-1)^n]$$

where m is an integer since $(n-2)!$ is divisible by $k!$ for $k \leq n-2$. Hence

$$!n = n! + [(-1)^{n+1} + m](n-1)$$

or

$$!n \equiv n! \pmod{n-1}$$

Also solved by J. M. Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Richard T. J. Mahoney, Washington University; Charles F. Osgood, Haverford College, Pennsylvania; Charles F. Pinzka, University of Cincinnati; Dale Woods and Cosmel R. Davis (Jointly), Idaho State College; and the proposer.

Arithmetic and Harmonic Means

343. [May 1958] *Proposed by Grant Heck, student at Lebanon Valley College, Pennsylvania.*

An old Hall and Knight problem reads, "If a is one of the 19 arithmetic means inserted between 2 and 3 and h is the corresponding harmonic mean, show that $a = 5 - 6/h$." Generalize this for any interval and for any number of inserted means and show that the correspondence is independent of the number of inserted means.

Solution by Monte Dernham, San Francisco, California. Regard a as the nth of m arithmetic means inserted between A and B, and h as the nth of m harmonic means. Then, it is readily found,

$$a = \frac{A(m+1) + n(B-A)}{m+1} \tag{1}$$

$$h = \frac{AB(m+1)}{B(m+1) - n(B-A)} \tag{2}$$

The desired generalization is given by

$$a = A + B - AB/h, \tag{3}$$

which reduces to an identity when a and h, respectively, are replaced by the right side of (1) and of (2). Since m and n have been eliminated from (3), the correspondence is "independent of the number of inserted means."

As a somewhat trivial but interesting corollary, it follows that, if a is any mean whatsoever of an arithmetic progression, viz., $A, \cdots, a, \cdots, B$, then, without regard to which mean or to the number of means inserted, the corresponding harmonic mean, h, may be found at once:

$$(A \neq 0,\ B \neq 0)$$
$$h = AB/(A+B-a). \tag{4}$$

Also solved by William E. F. Appuhn, St. John's University, New York; Felix A. Beiner, Western Electric Co., Cicero, Illinois; George M. Bergman, Stuyvesant High School, New York; J. M. Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; B. K. Gold, Los Angeles City College; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Herbert R. Leifer, Pittsburgh, Pennsylvania; Ralph L. London, Washington and Jefferson College, Pennsylvania; Michael J. Pascual, Burbank, California; Charles F. Pinzka, University of Cincinnati, Frank W. Saunders, Coker College, South Carolina; Sister M. Stephanie, Georgian Court College, New Jersey; C. W. Trigg, Los Angeles City College; and the proposer.

An Irrational Polynomial

344. [May 1958] *Proposed by James McCawley Jr., Chicago, Illinois.*

Let $f(x)$ be a polynomial whose coefficients are integers. If the leading coefficient and constant term and an odd number of the remaining coefficients are odd, prove that the equation $f(x) = 0$ has no rational root.

Solution by Joseph D. E. Konhauser, Haller Raymond and Brown, Inc., State College, Pennsylvania. Suppose $f(p/q) = 0$, where $(p, q) = 1$. Now p and q must divide the constant term and the leading coefficient, respectively, hence p and q must be odd. If n is the degree of $f(x)$, then $q^n f(p/q)$ must be zero, but this is impossible since $q^n f(p/q)$ is a sum of an odd number of odd numbers. Hence $f(x) = 0$ has no rational roots.

Also solved by William E. F. Appuhn, St. John's University, New York; George Bergman, Stuyvesant High School, New York; Richard T. J. Mahoney, Washington University; M. Morduchow, Polytechnic Institute of Brooklyn; Charles F. Osgood, Haverford College, Pennsylvania; F. D. Parker, University of Alaska; Michael J. Pascual, Burbank, California; Charles F. Pinzka, University of Cincinnati; and the proposer.

Relationship of Sums

345. [May 1958] *Proposed by D. A. Breault, Sylvania Electric Co., Waltham,*

Massachusetts.

Given that $\sum_{n=1}^{\infty} \frac{1}{n^3} = s$ prove the following:

a) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = 3/4\, s$$

b) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = 7/8\, s$$

I. *Solution by William M. Sanders, Mississippi Southern College.* Now

$$\frac{s}{4} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{4n^3} = \sum_{n=1}^{\infty} \frac{2}{(2n)^3}.$$

Subtracting this equation from the equation defining s gives a). Addition of the equation defining s to the equation a) results in a series involving only odd arguments. That is,

$$s + \frac{3s}{4} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

Consequently

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = \frac{7s}{8}.$$

II. *Generalization by Charles F. Pinzka, University of Cincinnati.* The results can be easily generalized to sums $s_k = \sum_{n=1}^{\infty} \frac{1}{n^k}$. Since the sum of the even-numbered terms is seen to be $s_k/2^k$, we have

(a) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^k} = s_k - 2s_k/2^k = (2^{k-1}-1)s_k/2^{k-1}$$

and

(b) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} = s_k - s_k/2^k = (2^k-1)s_k/2^k .$$

The obvious contradictions when $k = 1$ provide easy proofs that the harmonic series diverges.

Also solved by William E. F. Appuhn, St. John's University, New York; Feliz A. Beiner, Western Electric Co., Cicero, Illinois; George Bergman, Stuyvesant High School, New York; John L. Brown, Jr., Pennsylvania State University; J. M. Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; J. M. Howell, Los Angeles City College; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Richard T. J. Mahoney, Washington University; M. Morduchow, Polytechnic Institute of Brooklyn; Charles F. Osgood, Haverford College, Pennsylvania; F. D. Parker, University of Alaska; Michael J. Pascual, Burbank, California; William Squire, Bell Aircraft Corp., Buffalo, New York; John L. Wulff, Sacramento State College, California; and the proposer.

Erich Michalup, Caracas, Venezuela pointed out that the problem is found in Kowalewski's *Die Komplexen Veraenderlichen und ihre Funktionen*, Leipzig, 1923, page 129 substituting $s = 8J/7$.

Nested Radicals

346. [May 1958] *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India.*

Find the value of the expression

$$\sqrt[3]{11 + 4\sqrt[3]{14 + 10\sqrt[3]{17 + 18\sqrt[3]{(\cdots)}}}}$$

Solution by Edgar Karst, Endicott, New York. Obviously 11, 14, 17, ... have the sequence $8 + 3n$, and 4, 10, 18, ... have the sequence $n^2 + 3n$. The first integer cube above 11 is 27. To get 27 under the first cube root, there has to be $4^3 = 64$ under the second one, since $11 + 16 = 27$. But with $64 = 14 + 50$ under the second cube root, under the third one has to be $5^3 = 125$. Under the fourth one will be 6^3, and so on. Therefore, the value of the expression above is 3.

Also solved by the proposer.

Binomial Coefficients

347. [May 1958] *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

Let $\binom{n}{s}$ by the $(s+1)$ st binomial coefficient of order n. Prove that

$$\binom{n}{1}1^n - \binom{n}{2}2^n + \binom{n}{3}3^n - \cdots (-1)^{n+1}\binom{n}{n}n^n = (-1)^{n+1}n!$$

Solution by R. V. Parker, Bressingham, Diss, Norfolk, England. Let

$\Delta u_x = u_{x+1} - u_x$, and $Eu_x = u_{x+1}$. Then we have the well-known symbolical identity: $\Delta \equiv E - 1$. Thus

$$\Delta^n u_x = (E-1)^n u_x = \sum_{t=0}^{n} (-1)^t \binom{n}{t} E^{n-t} u_x$$

$$= \sum_{t=0}^{n} (-1)^t \binom{n}{t} u_{x+n-t} \qquad \text{(i)}$$

If, in (i), we put $u_x = [x^n]_{x=0}$, we obtain the nth difference of zero, which is known to be equal to factorial n.

$$\Delta^n 0^n = \sum_{t=0}^{n} (-1)^t \binom{n}{t} (n-t)^n = n! \qquad \text{(ii)}$$

In (ii) the coefficient of r^n may be obtained by putting $n-r$ for t or

$$(-1)^{n-r} \binom{n}{n-r} \qquad \text{(iii)}$$

Now Piza's statement may be expressed as

$$n! = (-1)^{n+1} \sum_{t=0}^{n} (-1)^{t+1} \binom{n}{t} t^n$$

$$= \sum_{t=0}^{n} (-1)^{n+t} \binom{n}{t} t^n \qquad \text{(iv)}$$

since the first term of the expansion of (iv) equals zero. In (iv) the coefficient of r^n is

$$(-1)^{n+r} \binom{n}{r} \qquad \text{(v)}$$

But (v) is equal to (iii), since $(-1)^{n-r} = (-1)^{n+r}$ and $\binom{n}{n-r} = \binom{n}{r}$. Therefore the expansions of (ii) and (iv) are identical, and since (ii) is correct, then so is (iv), hence Piza's proposition.

Also solved by J. M. Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; and Charles F. Pinzka, University of Cincinnati.

D. A. Breault, F. D. Parker and C. F. Pinzka referred this problem to problems 3625 (1934, p. 454), 4183 (1947, p. 235), E1253 (1957, p. 594) of the *American Mathematical Monthly* and in the Otto Dunkel Memorial Problem Book.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 236. Find the sum of the series

$$s = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots + ax^n + bx^{n+1} + (a+b)x^{n+2} + \cdots$$

for $|x| < 1$. [*Submitted by M. S. Klamkin*]

Q 237. Show that

$$\begin{vmatrix} x-2 & x-3 & x-4 \\ x+1 & x-1 & x-3 \\ x-4 & x-7 & x-10 \end{vmatrix} \equiv 0$$

[*Submitted by D. L. Silverman*]

Q 238. The sides of a triangle are $\sqrt{10}$, $\sqrt{5}$ and $\sqrt{13}$. Show that the area is exactly 3.5 square units. [*Submitted by Norman Anning*]

Q 239. A boy sent to buy \$1.00 worth of stamps asked for some two-cent stamps, ten times as many one-cent stamps and the rest in five-cent stamps. How many of each did he receive? [*Submitted by C. W. Trigg*]

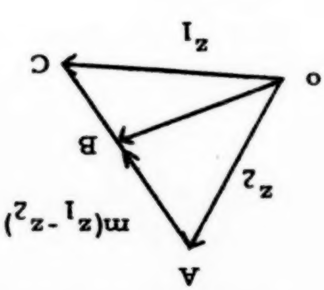
Q 240. Show that

$$mz_1 + (1-m)z_2 \leqq \max\,[|z_1|, |z_2|]$$

where $0 \leqq m \leqq 1$. [*Submitted by M. S. Klamkin*]

Answers

A 240. Proof:



Geometrically it follows that $\overline{OB} \leqq \max\,[\overline{OA}, \overline{OC}]$

A 236.

$$S = 1 + x + 2x^2 + 3x^3 + \cdots + ax^n + bx^{n+1} + (a+b)x^{n+2} + \cdots$$

$$xS = \quad x + x^2 + 2x^3 + \cdots$$

$$x^2S = \quad x^2 + x^3 + \cdots$$

so

$$S(1-x-x^2) = 1$$

or

$$S = 1/(1-x-x^2)$$

A 237. If $x = 1$, the first and third rows are proportional, and the determinant vanishes. Also, if $x = 5$, the first and second rows are proportional, if $x = -11$, the second and third rows are proportional, and if $x = -3$, the third row is proportional to the sum of the first two. Thus, the polynomial expansion has four distinct zeros, and since its degree is not greater than 3, it must be identically zero.

A 238. Examine the triangle whose vertices are (0, 0), (3, 1) and (2, 3). The area is 9–3–1–1.5 square units. (Alternate solution)

$$16\Delta^2 = \Sigma(2b^2c^2-a^4) = 130 + 260 + 100 - 100 - 25 - 169 = 196,$$

so $4\Delta^2 = 49$ or $\Delta = 3.5$.

A 239. The one- and two-cent stamps were purchased in twelve-cent lots, and the amount spent for them had to be a multiple of 5, or 60 cents. Hence, 5 one-cent, 50 two-cent, and 8 five-cent stamps were purchased.

PERKINS ABACUS SERVICE

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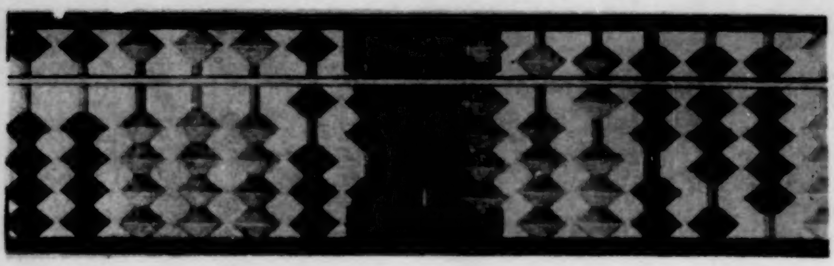

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